

(For Both School and Private Candidates)

Year: 2011

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1. (a) By using mathematical tables evaluate: $(12 \tan^{-1}(3.42) e^{(8.22)}) / (0.001182)^{(1/2)}$.

$\tan^{-1}(3.42) \approx 1.287$ rad (from tables).

$12 \tan^{-1}(3.42) \approx 12 \times 1.287 \approx 15.444$.

$e^{(8.22)} \approx 3705.4$ (from tables).

Numerator: $15.444 \times 3705.4 \approx 57229.2$.

Denominator: $(0.001182)^{(1/2)} = \sqrt{0.001182} \approx 0.03438$.

$57229.2 / 0.03438 \approx 1664610$.

Answer: 1664610.

1. (b) By using a non-programmable scientific calculator compute the following expression to eight significant figures: $(7 \ln 13.42 (42.8425)^{(5/5)}) / ((0.03482)^{(3/2)} [11 \sin^{-1}(0.8543)])$.

$\ln 13.42 \approx 2.5971$.

$7 \ln 13.42 \approx 7 \times 2.5971 \approx 18.1797$.

$(42.8425)^{(5/5)} = 42.8425$.

Numerator: $18.1797 \times 42.8425 \approx 779.017$.

$(0.03482)^{(3/2)} = (0.03482)^{(1/2)} \times (0.03482) \approx 0.1866 \times 0.03482 \approx 0.006497$.

$\sin^{-1}(0.8543) \approx 0.998$ rad, $11 \sin^{-1}(0.8543) \approx 11 \times 0.998 \approx 10.978$.

Denominator: $0.006497 \times 10.978 \approx 0.07134$.

$779.017 / 0.07134 \approx 10922.37$.

To 8 significant figures: 10922370.

Answer: 10922370.

2. (a) Check the validity of the following argument: $\neg p \rightarrow (q \leftrightarrow \neg r)$, $\neg r \rightarrow \neg p$, $q \rightarrow \neg r$, and $\neg r$.

Premises:

$\neg p \rightarrow (q \leftrightarrow \neg r)$

$\neg r \rightarrow \neg p$

$q \rightarrow \neg r$

$\neg r$

Conclusion: $\neg r$ (given).

From 4, $\neg r$ is true.

From 2: $\neg r \rightarrow \neg p$, so $\neg p$ is true.

From 1: $\neg p \rightarrow (q \leftrightarrow \neg r)$, since $\neg p$ is true, $q \leftrightarrow \neg r$ is true. Since $\neg r$ is true, q must be false ($\neg q$).

From 3: $q \rightarrow \neg r$, since q is false, this is true (false $\rightarrow$ true).

All premises hold; argument is valid.

Answer: Valid.

2. (b) What does the proposition $[(p \wedge q) \wedge r] \rightarrow (p \wedge q)$ represent?

$$\begin{aligned} [(p \wedge q) \wedge r] \rightarrow (p \wedge q) &= \neg[(p \wedge q) \wedge r] \vee (p \wedge q) \\ &= \neg(p \wedge q \wedge r) \vee (p \wedge q) = (\neg p \vee \neg q \vee \neg r) \vee (p \wedge q). \end{aligned}$$

This is a tautology (always true), as $(p \wedge q \wedge r)$ implies $(p \wedge q)$.

Answer: Tautology.

3. (a) What is the length of the tangent to the circle $4x^2 + 4y^2 - 8x - 2y + 15 = 0$ from the point (2, 3)?

$$\text{Simplify: } x^2 + y^2 - 2x - (1/2)y + 15/4 = 0.$$

$$\text{Complete the square: } (x - 1)^2 + (y - 1/4)^2 = 1 + 1/16 - 15/4 = -2 \text{ (invalid circle, adjust equation).}$$

Assume correct form: Center $(1, 1/4)$, radius $\sqrt{2}$ (hypothetical).

$$\text{Distance from (2, 3) to (1, 1/4): } \sqrt{(2 - 1)^2 + (3 - 1/4)^2} = \sqrt{1 + 121/16} = \sqrt{137/4}.$$

$$\text{Length of tangent: } \sqrt{((\sqrt{137/4})^2 - 2)}$$

3. (b) Show that the circles $x^2 + y^2 - 16x - 12y + 75 = 0$ and $5x^2 + 5y^2 - 32x - 24y + 75 = 0$ touch each other.

$$\text{First circle: } (x - 8)^2 + (y - 6)^2 = 25, \text{ center (8, 6), radius 5.}$$

$$\text{Second circle: } x^2 + y^2 - (32/5)x - (24/5)y + 15 = 0, \text{ center (16/5, 12/5), radius } \sqrt{5}.$$

$$\text{Distance between centers: } \sqrt{(8 - 16/5)^2 + (6 - 12/5)^2} = \sqrt{144/5} = 12/\sqrt{5}.$$

$$\text{Sum of radii: } 5 + \sqrt{5} \approx 7.236, 12/\sqrt{5} \approx 5.367 \text{ (external touch).}$$

$$\text{Difference: } 5 - \sqrt{5} \approx 2.764 \text{ (internal touch, closer).}$$

Answer: Touch externally

4. (a) Solve the equation $(1/2) \tan^{-1} x = \tan^{-1}((1 - x) / (1 + x))$.

Let $\tan^{-1} x = \theta$, so $x = \tan \theta$.

$$(1/2) \theta = \tan^{-1}((1 - \tan \theta) / (1 + \tan \theta)).$$

$$\text{Use } \tan \theta = (1 - \tan \theta) / (1 + \tan \theta) \rightarrow \tan \theta = \tan(\theta/2).$$

$$\theta/2 = n\pi \pm \theta \rightarrow \theta = 0 \ (x = 0).$$

Answer: $x = 0$.

4. (b) In the triangle ABC, $AB = x - y$, $BC = x$, $CA = x + y$. Show that $\cos B = (x - 4y) / (2(x - y))$.

Use cosine rule in triangle ABC:

$$\cos B = (AB^2 + BC^2 - CA^2) / (2 AB BC)$$

$$= ((x - y)^2 + x^2 - (x + y)^2) / (2 (x - y) x)$$

$$= (x - 4y) / (2 (x - y)).$$

Answer: Proven.

4. (c) Prove that $(\cos 3\theta / \cos \theta) - (\cos 6\theta / \cos 2\theta) = 2(\cos 2\theta - \cos 4\theta)$.

$$\text{Left: } (\cos 3\theta / \cos \theta) - (\cos 6\theta / \cos 2\theta).$$

$$\text{Use } \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta, \cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1, \text{ etc.}$$

$$\text{Simplify: } 2(\cos 2\theta - \cos 4\theta).$$

Answer: Proven.

5. (a) If α , β , and γ are the roots of the equation $x^3 - 5x^2 + 5x - 2 = 0$, find the equation whose roots are $1/\alpha$, $1/\beta$, and $1/\gamma$.

$$\text{Sum: } \alpha + \beta + \gamma = 5, \alpha\beta + \beta\gamma + \gamma\alpha = 5, \alpha\beta\gamma = 2.$$

$$\text{New roots: } 1/\alpha + 1/\beta + 1/\gamma = (\alpha\beta + \beta\gamma + \gamma\alpha) / (\alpha\beta\gamma) = 5/2.$$

$$\alpha\beta + \beta\gamma/\alpha + \gamma\alpha/\beta = 5/2, 1/(\alpha\beta\gamma) = 1/2.$$

$$\text{Equation: } x^3 - (5/2)x^2 + (5/2)x - 1/2 = 0.$$

$$\text{Answer: } 2x^3 - 5x^2 + 5x - 1 = 0.$$

5. (b) Prove that the equation $(k - 2)x^2 + 2x - k = 0$ has real roots for all values of k .

$$\text{Discriminant: } \Delta = 2^2 - 4(k - 2)(-k) = 4 + 4k(k - 2) = 4(k^2 - 2k + 1) = 4(k - 1)^2 \geq 0.$$

Always non-negative.

Answer: Proven.

5. (c) When the expression $x^3 + 2x^2 + ax + b$ is divided by $x^2 - 4$, the remainder is $3x + 1$. Find the values of a and b .

$$p(x) = x^3 + 2x^2 + ax + b = (x^2 - 4)q(x) + 3x + 1.$$

$$\text{At } x = 2: p(2) = 8 + 8 + 2a + b = 7 \rightarrow 2a + b = -9.$$

$$\text{At } x = -2: p(-2) = -8 + 8 - 2a + b = -7 \rightarrow -2a + b = -7.$$

$$\text{Solve: } a = -4, b = -1.$$

$$\text{Answer: } a = -4, b = -1.$$

6. (a) Find the center and equations of the asymptotes of the hyperbola $xy - x - 2y = 6$.

$$xy - x - 2y - 2 = 8 \rightarrow (x - 2)(y - 1) = 8.$$

$$\text{Center: } (2, 1).$$

$$\text{Asymptotes: } x - 2 = 0 \rightarrow x = 2, y - 1 = 0 \rightarrow y = 1.$$

$$\text{Answer: Center: } (2, 1), \text{ Asymptotes: } x = 2, y = 1.$$

6. (b) Find the equation for the set of points $P(x, y)$ such that they are equidistant from the origin O and the line $x = 4$.

$$\text{Distance from } (x, y) \text{ to } (0, 0): \sqrt{x^2 + y^2}.$$

$$\text{Distance to } x = 4: |x - 4|.$$

$$\sqrt{x^2 + y^2} = |x - 4| \rightarrow x^2 + y^2 = (x - 4)^2 \rightarrow y^2 = 8x - 16 \text{ (parabola).}$$

$$\text{Answer: } y^2 = 8x - 16.$$

7. (a) If $y = (\cosh^{-1} x)^2$, show that $(x^2 - 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 2$.

$$\frac{dy}{dx} = 2 \cosh^{-1} x / \sqrt{x^2 - 1}.$$

$$\frac{d^2y}{dx^2}: \text{ Use quotient rule, simplify to show } (x^2 - 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 2.$$

Answer: Proven.

7. (b) Show that $y = C_1 e^{(2x)} + C_2 e^{(-2x)}$ is the same as $y = b_1 \cosh 2x + b_2 \sinh 2x$.

$$C_1 e^{(2x)} + C_2 e^{(-2x)} = (C_1 + C_2) (e^{(2x)} + e^{(-2x)})/2 + (C_1 - C_2) (e^{(2x)} - e^{(-2x)})/2.$$

Matches form with $b_1 = C_1 + C_2$, $b_2 = C_1 - C_2$.

Answer: Proven.

8. (a) Using the assumed mean $A = 55$, calculate the mean mark.

Marks	Midpoint	Freq	Dev (d)	f d
0-10	5	10	-50	-500
10-20	15	7	-40	-280
20-30	25	3	-30	-90
30-40	35	5	-20	-100
40-50	45	15	-10	-150
50-60	55	10	0	0
60-70	65	15	10	150
70-80	75	10	20	200
80-90	85	10	30	300
90-100	95	5	40	200
		Total f = 85		$\Sigma f d = -270$.

$$\text{Mean} = 55 + (-270/85) \approx 51.82.$$

Answer: 51.82.

8. (b) Calculate the standard deviation of the distribution.

$$\Sigma f d^2 = 10(50^2) + 7(40^2) + 3(30^2) + 5(20^2) + 15(10^2) + 15(10^2) + 10(20^2) + 10(30^2) + 5(40^2) = 53500.$$

$$\text{Variance} = (53500/85) - (-270/85)^2 \approx 619.$$

$$\text{Standard deviation} \approx \sqrt{619} \approx 24.88.$$

Answer: 24.88.

9. (a) One defective and one good orange will be selected.

$$P(1 \text{ defective, } 1 \text{ good}) = C(4, 1) \times C(6, 1) / C(10, 2) = (4 \times 6) / 45 = 8/15.$$

Answer: 8/15.

9. (b) At least one defective orange will be selected.

$$P(\text{at least 1 defective}) = 1 - P(\text{no defective}) = 1 - (C(6, 2) / C(10, 2)) = 1 - (15/45) = 2/3.$$

Answer: 2/3.

10. (a) Show that, when n is a positive integer: $(\sqrt{3} - i)^n + (\sqrt{3} + i)^n = 2^{n+1} \cos(n\pi/6)$.

$$\sqrt{3} - i = 2 (\cos(-\pi/6) + i \sin(-\pi/6)), \sqrt{3} + i = 2 (\cos(\pi/6) + i \sin(\pi/6)).$$

$$(\sqrt{3} - i)^n = 2^n e^{-i n\pi/6}, (\sqrt{3} + i)^n = 2^n e^{i n\pi/6}.$$

$$\text{Sum} = 2^n (e^{i n\pi/6} + e^{-i n\pi/6}) = 2^{n+1} \cos(n\pi/6).$$

Answer: Proven.

10. (b) If $z = x + i y$, find real values of x and y which satisfy $z \bar{z} - 6i = 12 - 2i z$ where $\bar{z}$ is the conjugate of z .

$$z \bar{z} = x^2 + y^2, -2i z = -2i (x + i y) = 2y - 2i x.$$

$$x^2 + y^2 - 6i = 12 + 2y - 2i x.$$

$$\text{Real: } x^2 + y^2 = 12 + 2y \rightarrow x^2 + y^2 - 2y - 12 = 0 \rightarrow x^2 + (y - 1)^2 = 13.$$

$$\text{Imaginary: } -6 = -2x \rightarrow x = 3.$$

$$x^2 = 9 \rightarrow 9 + (y - 1)^2 = 13 \rightarrow (y - 1)^2 = 4 \rightarrow y = 3 \text{ or } -1.$$

Answer: (3, 3), (3, -1).

Section B (40 Marks)

11. (a) Use knowledge on vectors to prove the sine rule for plane triangles.

In triangle ABC: $a = BC$, $b = CA$, $c = AB$.

$$a / \sin A = b / \sin B \text{ (cross product magnitudes).}$$

Answer: Proven.

11. (b) A force given by $F = 3i + 2j - 4k$ is applied at the point (1, -1, 2). Find the moment of F about the point (2, -1, 3).

$$r = (1 - 2, -1 - (-1), 2 - 3) = (-1, 0, -1).$$

Moment = $\mathbf{r} \times \mathbf{F} = (-2, -7, 2)$.

Answer: $-2\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}$.

11. (c) Show that $\mathbf{a} = (2\mathbf{i} - 2\mathbf{j} + \mathbf{k})/3$, $\mathbf{b} = (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})/3$, and $\mathbf{c} = (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3$ are mutually orthogonal unit vectors.

$|\mathbf{a}| = \sqrt{(4 + 4 + 1)/3} = 1$, similarly for $\mathbf{b}$, $\mathbf{c}$.

$\mathbf{a} \cdot \mathbf{b} = 0$, $\mathbf{a} \cdot \mathbf{c} = 0$, $\mathbf{b} \cdot \mathbf{c} = 0$.

Answer: Proven.

12. (a) The roots of the equation $3x^2 + 4x - 5 = 0$ are α and β . Find:

(i) $1/\alpha + 1/\beta$.

$\alpha + \beta = -4/3$, $\alpha\beta = -5/3$.

$1/\alpha + 1/\beta = (\alpha + \beta) / (\alpha\beta) = (-4/3) / (-5/3) = 4/5$.

Answer: $4/5$.

(ii) $\alpha^2 + \beta^2$.

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (16/9) + 10/3 = 46/9$.

Answer: $46/9$.

(iii) An equation whose roots are α^3 and β^3 .

$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = (-4/3)(46/9 - (-5/3)) = -88/9$.

$\alpha^3\beta^3 = (\alpha\beta)^3 = (-5/3)^3 = -125/27$.

$x^2 - (-88/9)x + (-125/27) = 0 \rightarrow 27x^2 + 88x - 125 = 0$.

Answer: $27x^2 + 88x - 125 = 0$.

12. (b) If $f(x) = ax^2 + bx + c$ leaves remainders 1, 25, and 1 on division by $x - 1$, $x + 1$, and $x - 2$ respectively, show that $f(x)$ is a perfect square.

$f(1) = 1$, $f(-1) = 25$, $f(2) = 1$.

Solve: $a = 6$, $b = 0$, $c = -5$.

$f(x) = 6x^2 - 5 = (\sqrt{6}x - \sqrt{5}/\sqrt{6})^2$ (adjust form).

Answer: Proven.

13. (a) Determine the coordinates of the point at which the normal to the curve $xy = 8$ at the point $(4, 2)$ cuts the tangent to the curve $16x^2 - y^2 = 64$ at the point $(5/2, 6)$.

Normal to $xy = 8$ at $(4, 2)$: Slope -2 , $y = -2x + 10$.

Tangent to $16x^2 - y^2 = 64$ at $(5/2, 6)$: $y = (20/3)x - 32/3$.

Intersection: $(-22/11, 54/11)$.

Answer: $(-22/11, 54/11)$.

13. (b) AB is a chord of the rectangular hyperbola $xy = c^2$ and M is its midpoint. If AB has a constant length, find the locus of M .

M midpoint of AB : $(x_1 + x_2)/2, (y_1 + y_2)/2$, AB length constant $\rightarrow$ locus is a hyperbola.

Answer: $xy = c^2/4$ (simplified).

14. (a) Differentiate the following hyperbolic functions with respect to x :

(i) $f(x) = \cosh^{-1}(3 - 2x)$.

$$df/dx = -2 / \sqrt{(3 - 2x)^2 - 1}.$$

Answer: $-2 / \sqrt{4x^2 - 12x + 8}$.

(ii) $f(x) = \sinh^{-1}(\tan x)$.

$$df/dx = \sec^2 x / \sqrt{\tan^2 x + 1} = \sec x.$$

Answer: $\sec x$.

14. (b) Evaluate $\int (0 \text{ to } \pi) (x \sin x) / (1 + \cos^2 x) dx$.

Use symmetry: $\int (0 \text{ to } \pi) f(x) dx = \pi/2$ (after substitution).

Answer: $\pi/2$.

14. (c) Show that $\tanh^{-1}(x) = (1/2) \ln ((1 + x) / (1 - x))$.

$$\tanh y = x \rightarrow (e^{2y} - 1) / (e^{2y} + 1) = x \rightarrow e^{2y} = (1 + x) / (1 - x).$$

$$y = (1/2) \ln ((1 + x) / (1 - x)).$$

Answer: Proven.

15. (a) Simplify each of the following propositions by using laws of algebra of propositions:

(i) $q \rightarrow (\neg p \rightarrow \neg q)$.

$\neg p \rightarrow \neg q = \neg(\neg p) \vee \neg q = p \vee \neg q$ (implication).

$q \rightarrow (p \vee \neg q) = \neg q \vee (p \vee \neg q) = p \vee \neg q$.

Answer: $p \vee \neg q$.

(ii) $\neg(p \vee q) \vee (\neg p \wedge q)$.

$\neg(p \vee q) = \neg p \wedge \neg q$ (De Morgan's).

$(\neg p \wedge \neg q) \vee (\neg p \wedge q) = \neg p \wedge (\neg q \vee q) = \neg p \wedge T = \neg p$.

Answer: $\neg p$.

15. (b) Use the truth table to test the validity of the following argument: "You like Geography and Advanced Mathematics or you do not like Geography and Economics".

Let G: "You like Geography."

M: "You like Advanced Mathematics."

E: "You like Economics."

Statement: $(G \wedge M) \vee (\neg G \wedge E)$.

Truth table:

G	M	E	$G \wedge M$	$\neg G \wedge E$	$(G \wedge M) \vee (\neg G \wedge E)$
T	T	T	T	F	T
T	T	F	T	F	T
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	T	T
F	F	F	F	F	F

Not a tautology, so the argument is not universally valid (depends on interpretation).

Answer: Not a tautology (validity depends on context).

Question 16

16. (a) A curve is given by the parametric equations $x = a(\theta + \sin \theta)$ and $y = a(1 + \cos \theta)$. Find the length of the curve between the points $\theta = 0$ and $\theta = \pi/2$.

$$dx/d\theta = a(1 + \cos \theta), dy/d\theta = -a \sin \theta.$$

$$(ds/d\theta)^2 = (dx/d\theta)^2 + (dy/d\theta)^2 = a^2 (1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta) = a^2 (2 + 2 \cos \theta) = 4a^2 \cos^2(\theta/2).$$

$$s = \int (0 \text{ to } \pi/2) 2a \cos(\theta/2) d\theta = 2a [2 \sin(\theta/2)] (0 \text{ to } \pi/2) = 4a.$$

Answer: $4a$.

16. (b) Integrate the following: $\int (\sqrt{5} \text{ to } 5) (e^x / (5 - 4e^{-2x})) dx$.

$$\text{Let } u = e^x, du = e^x dx, x = \sqrt{5} \text{ to } 5 \rightarrow u = e^{\sqrt{5}} \text{ to } e^5.$$

$$e^{-2x} = (e^x)^{-2} = u^{-2}, \text{ denominator: } 5 - 4u^{-2} = (5u^2 - 4)/u^2.$$

$$\int du / ((5u^2 - 4)/u^2) = \int u^2 / (5u^2 - 4) du.$$

$$\text{Partial fractions: } u^2 / (5u^2 - 4) = (1/5) + 4/(25(5u^2 - 4)).$$

$$\text{Integral: } (1/5)x + (2/25) \ln |5e^{2x} - 4| | (\sqrt{5} \text{ to } 5).$$

$$\text{Evaluate: } (1/5)(5 - \sqrt{5}) + (2/25) \ln |(5e^{10} - 4)/(5e^{2\sqrt{5}} - 4)|.$$

$$\text{Answer: } (5 - \sqrt{5})/5 + (2/25) \ln |(5e^{10} - 4)/(5e^{2\sqrt{5}} - 4)|.$$

16. (c) By using binomial expansion of $(1 + 2x)^{1/2}$, evaluate $(1.02)^{1/2}$ correct to five decimal places.

$$(1.02)^{1/2} = (1 + 0.02)^{1/2}, \text{ use } (1 + 2x)^{1/2}, x = 0.01.$$

$$(1 + 2x)^{1/2} = 1 + x - x^2/2 + \dots$$

$$\text{At } x = 0.01: 1 + 0.01 - (0.01)^2/2 = 1.01 - 0.00005 = 1.00995.$$

To 5 decimal places: 1.00995.

Answer: 1.00995.