

THE UNITED REPUBLIC OF TANZANIA
NATIONAL EXAMINATIONS COUNCIL
ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION
142/2 **ADVANCED MATHEMATICS 2**

(For Both School and Private Candidates)

Time: 3 Hours

ANSWERS

Year: 2016

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

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1. (a)(i) Use De Moivre's theorem to find the value of $[1 + i]^4$

De Moivre's theorem: $(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta)$

Rewrite $1 + i$ in polar form:

$$|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\arg(1 + i) = \tan^{-1}(1/1) = \pi/4$$

$$1 + i = \sqrt{2} [\cos(\pi/4) + i \sin(\pi/4)]$$

$$[1 + i]^4 = [\sqrt{2} (\cos(\pi/4) + i \sin(\pi/4))]^4$$

$$= (\sqrt{2})^4 [\cos(4 \pi/4) + i \sin(4 \pi/4)]$$

$$= 4 [\cos(\pi) + i \sin(\pi)]$$

$$= 4 [-1 + i0] = -4$$

Answer: -4

(a)(ii) Use mathematical induction to prove $[r(\cos\theta + i \sin\theta)]^n = r^n (\cos(n\theta) + i \sin(n\theta))$

Base case ($n = 1$):

$$[r(\cos\theta + i \sin\theta)]^1 = r^1 (\cos(1 \theta) + i \sin(1\theta))$$

True.

Assume true for $n = k$:

$$[r(\cos\theta + i \sin\theta)]^k = r^k (\cos(k\theta) + i \sin(k\theta))$$

Prove for $n = k + 1$:

$$[r(\cos\theta + i \sin\theta)]^{(k+1)} = [r(\cos\theta + i \sin\theta)]^k \times [r(\cos\theta + i \sin\theta)]$$

$$= [r^k (\cos(k\theta) + i \sin(k\theta))] \times [r(\cos\theta + i \sin\theta)]$$

$$= r^{(k+1)} [(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta) + i (\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta)]$$

$$= r^{(k+1)} [\cos(k\theta + \theta) + i \sin(k\theta + \theta)]$$

$$= r^{(k+1)} [\cos((k+1)\theta) + i \sin((k+1)\theta)]$$

True for $n = k + 1$. By induction, the statement holds for all positive integers n .

(b)(i) If $\arg((z - 1)/(z + 1)) = \pi/4$ and $z = x + iy$, find the locus of z in an Argand diagram

Let $z = x + iy$. Then:

$$(z - 1)/(z + 1) = (x + iy - 1)/(x + iy + 1)$$

$$= [(x - 1) + iy]/[(x + 1) + iy]$$

Rationalize: Multiply numerator and denominator by conjugate $(x + 1) - iy$

$$\text{Numerator: } [(x - 1) + iy][(x + 1) - iy] = (x - 1)(x + 1) - (x - 1)iy + (x + 1)iy - iy * iy$$

$$= (x^2 - 1) + y^2$$

$$\text{Denominator: } (x + 1)^2 + y^2$$

$$\text{So, } (z - 1)/(z + 1) = [(x^2 + y^2 - 1) + i(2y)]/[(x + 1)^2 + y^2]$$

$$\text{Argument: } \tan^{-1}(2y/(x^2 + y^2 - 1)) = \pi/4$$

$$\Rightarrow \tan(\pi/4) = 1$$

$$\Rightarrow 2y/(x^2 + y^2 - 1) = 1$$

$$\Rightarrow 2y = x^2 + y^2 - 1$$

$$\Rightarrow x^2 + y^2 - 2y - 1 = 0$$

$$\Rightarrow x^2 + (y - 1)^2 = 2$$

Locus: Circle with center $(0, 1)$, radius $\sqrt{2}$.

(b)(ii) Solve the system $iz + w = 2i$, $z - iw = 2i$, where z and w are complex numbers

$$\text{From (1): } iz + w = 2i$$

$$\text{From (2): } z - iw = 2i$$

Multiply (2) by i :

$$i(z - iw) = i \times 2i$$

$$\Rightarrow iz + w = -2$$

$$\text{Now: } iz + w = 2i \quad (1)$$

$$iz + w = -2 \quad (3)$$

This is inconsistent unless $2i = -2$, which is false.

System has no solution.

(c) One root of $z^4 - 6z^3 + 23z^2 - 34z + 26 = 0$ is $1 + i$. Find the other roots

Given root: $1 + i$. Since coefficients are real, conjugate $1 - i$ is also a root.

Quadratic factor: $(z - (1 + i))(z - (1 - i)) = z^2 - (1 + i + 1 - i)z + (1 + i)(1 - i)$
 $= z^2 - 2z + 2$

Divide $z^4 - 6z^3 + 23z^2 - 34z + 26$ by $z^2 - 2z + 2$:

$$z^4 - 6z^3 + 23z^2 - 34z + 26 = (z^2 - 2z + 2)(z^2 - 4z + 13)$$

Solve $z^2 - 4z + 13 = 0$:

Discriminant: $(-4)^2 - 4 * 13 = 16 - 52 = -36$

$$z = (4 \pm \sqrt{-36})/2 = 2 \pm 3i$$

Roots: $1 + i, 1 - i, 2 + 3i, 2 - 3i$.

2. (a)(i) Draw the simplified electrical circuit for $[(p \wedge (p \vee q))] \wedge [q \rightarrow (p \wedge q)]$

Simplify $[(p \wedge (p \vee q))] \wedge [q \rightarrow (p \wedge q)]$:

$$p \wedge (p \vee q) = p \text{ (absorption law)}$$

$$q \rightarrow (p \wedge q) = \neg q \vee (p \wedge q)$$

$$= (\neg q \vee p) \wedge (\neg q \vee q) = \neg q \vee p$$

$$\text{So, expression} = p \wedge (\neg q \vee p) = p$$

Circuit: Single wire (p).

(a)(ii) Use truth table to show whether or not $p \rightarrow q$ and $\neg(p \wedge q)$ are logically equivalent

P	q	$p \rightarrow q$	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	T	F
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$p \rightarrow q \neq \neg(p \wedge q)$. Not logically equivalent.

(b)(i) Use truth table to test validity: If I am intelligent, then I will pass this examination. I am intelligent. Therefore I will pass this examination.

Let p: I am intelligent, q: I will pass this examination.

Premises: $p \rightarrow q$, p . Conclusion: q .

P	q	$p \rightarrow q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

When $p \rightarrow q$ and p are true, q is true. Argument is valid.

(b)(ii) Write converse, inverse, and contrapositive of "If Mathematics is interesting then Biology is boring and tough"

Statement: $p \rightarrow (q \wedge r)$, p : Mathematics is interesting, q : Biology is boring, r : Biology is tough.

Converse: $(q \wedge r) \rightarrow p$

If Biology is boring and tough, then Mathematics is interesting.

Inverse: $\neg p \rightarrow \neg(q \wedge r)$

If Mathematics is not interesting, then Biology is not boring or not tough.

Contrapositive: $\neg(q \wedge r) \rightarrow \neg p$

If Biology is not boring or not tough, then Mathematics is not interesting.

3. (a)(i) If position vectors OA , OB , OC are $OA = 2i - j + 3k$, $OB = 3i + 2j - 4k$, $OC = -i + 3j - 2k$, determine $AB \times BC$

$$AB = OB - OA = (3i + 2j - 4k) - (2i - j + 3k) = i + 3j - 7k$$

$$BC = OC - OB = (-i + 3j - 2k) - (3i + 2j - 4k) = -4i + j + 2k$$

$$AB \times BC = \begin{vmatrix} i & j & k \\ 1 & 3 & -7 \\ -4 & 1 & 2 \end{vmatrix}$$

$$= i(3 \cdot 2 - (-7) \cdot 1) - j(1 \cdot 2 - (-7) \cdot (-4)) + k(1 \cdot 1 - 3 \cdot (-4))$$

$$= i(6 + 7) - j(2 - 28) + k(1 + 12)$$

$$= i(13) - j(-26) + k(13)$$

$$= 13i + 26j + 13k$$

$$= 13i + 26j + 13k$$

(a)(ii) Find the exact value of the angle between AB and BC

$$|AB| = \sqrt{1^2 + 3^2 + (-7)^2} = \sqrt{59}$$

$$|BC| = \sqrt{(-4)^2 + 1^2 + 2^2} = \sqrt{21}$$

$$AB \cdot BC = 1(-4) + 3 \times 1 + (-7) \times 2 = -4 + 3 - 14 = -15$$

$$\cos\theta = (AB \cdot BC)/(|AB||BC|) = -15/(\sqrt{59} \times \sqrt{21})$$

$$\text{Exact value: } -15/\sqrt{1239}$$

(b)(i) If $a = 3i + 2j - 9k$, $b = i + j + 3k$, find value of k so that $a + kb$ is perpendicular to $a - b$

$$a + kb = (3 + k)i + (2 + k)j + (-9 + 3k)k$$

$$a - b = (3 - 1)i + (2 - 1)j + (-9 - 3)k = 2i + j - 12k$$

Dot product = 0:

$$(3 + k)2 + (2 + k)1 + (-9 + 3k)(-12) = 0$$

$$6 + 2k + 2 + k + 108 - 36k = 0$$

$$-33k + 116 = 0$$

$$k = 116/33$$

(b)(ii) Projection of a onto b in surd form

$$|a| = \sqrt{3^2 + 2^2 + (-9)^2} = \sqrt{94}$$

$$|b| = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}$$

$$a \cdot b = 3 \times 1 + 2 \times 1 + (-9) \times 3 = 3 + 2 - 27 = -22$$

$$\text{Projection of } a \text{ onto } b = (a \cdot b)/|b| = -22/\sqrt{11}$$

(c) Derive the cosine rule using vectors a and b

Let $c = a - b$.

$$c^2 = (a - b) \cdot (a - b) = a^2 + b^2 - 2a \cdot b$$

$$a^2 + b^2 - c^2 = 2a \cdot b$$

$$a \cdot b = |a||b|\cos\theta$$

$$\cos\theta = (a^2 + b^2 - c^2)/(2|a||b|)$$

Cosine rule: $\cos\theta = (a^2 + b^2 - c^2)/(2ab)$

4. (a)(i) Find the value of a if the 17th and 18th terms of the expansion $(2 + a)^n$ are equal

For $(2 + a)^n$, the general term is:

$$T(r+1) = C(n,r) \times 2^{n-r} \times a^r$$

$$17\text{th term } (r = 16): T(17) = C(n,16) \times 2^{n-16} \times a^{16}$$

$$18\text{th term } (r = 17): T(18) = C(n,17) \times 2^{n-17} \times a^{17}$$

Given $T(17) = T(18)$:

$$C(n,16) \times 2^{n-16} \times a^{16} = C(n,17) \times 2^{n-17} \times a^{17}$$

$$C(n,16)/C(n,17) \times 2^{n-16-(n-17)} = a^{17-16}$$

$$C(n,16)/C(n,17) = (n-17+1)/(17) = (n-16)/17$$

$$\text{So, } (n-16)/17 \times 2^1 = a$$

$$\Rightarrow a = 2(n-16)/17$$

We need n (not given). Assuming binomial context, if n is an integer, a depends on n . Without n , we leave:

$$a = 2(n-16)/17$$

(a)(ii) The roots of $x^3 + px^2 + qx + 30 = 0$ are in the ratio 2:3:5. Find p and q

Let roots be $2k, 3k, 5k$.

$$\text{Sum of roots: } 2k + 3k + 5k = 10k = -p$$

$$\text{Product of roots: } 2k \times 3k \times 5k = 30k^3 = -30$$

$$\text{Sum of pairwise products: } (2k)(3k) + (2k)(5k) + (3k)(5k) = 6k^2 + 10k^2 + 15k^2 = 31k^2 = q$$

$$\text{From product: } 30k^3 = -30$$

$$\Rightarrow k^3 = -1$$

$$\Rightarrow k = -1$$

$$\text{Sum of roots: } 10k = -10 = -p$$

$$\Rightarrow p = 10$$

Pairwise products: $31k^2 = 31(-1)^2 = 31 = q$

$\Rightarrow q = 31$

Answer: $p = 10, q = 31$

(b)(i) State the principle of mathematical induction as it is used in mathematics

Mathematical induction: To prove a statement $P(n)$ for all positive integers n :

Base case: Prove $P(1)$ is true.

Inductive step: Assume $P(k)$ is true for some $k \geq 1$ (inductive hypothesis), then prove $P(k+1)$ is true.

(b)(ii) Use mathematical induction to prove $\Sigma(3r-1) = n/2(3n+1)$, r from 1 to n

Base case ($n = 1$):

Left: $\Sigma(3r-1) = 3(1)-1 = 2$

Right: $1/2 \times (3(1) + 1) = 1/2 \times 4 = 2$

True.

Assume true for $n = k$:

$\Sigma(3r-1) = k/2 \times (3k+1)$

Prove for $n = k+1$:

$\Sigma(3r-1)$ from $r=1$ to $k+1 = [\Sigma(3r-1)$ from $r=1$ to $k] + (3(k+1)-1)$

$= k/2 \times (3k+1) + (3k+2)$

$= (3k^2 + k)/2 + (3k+2)$

$= (3k^2 + k + 6k + 4)/2$

$= (3k^2 + 7k + 4)/2$

$= (k+1)/2 \times (3k+4)$

$= (k+1)/2 \times (3(k+1) + 1)$

Matches the form. True for $n = k+1$. By induction, the statement holds for all positive integers n .

5. (a)(i) Solve $\tan^{-1}((x-1)/(x-2)) + \tan^{-1}((x+1)/(x+2)) = \pi/4$, leave answer in surd form

Let $A = \tan^{-1}((x-1)/(x-2))$, $B = \tan^{-1}((x+1)/(x+2))$.

Given: $A + B = \pi/4$

$$\Rightarrow \tan(A + B) = \tan(\pi/4) = 1$$

Using $\tan(A + B) = (\tan A + \tan B)/(1 - \tan A \tan B)$:

$$\tan A = (x-1)/(x-2), \tan B = (x+1)/(x+2)$$

$$\text{Numerator: } (x-1)/(x-2) + (x+1)/(x+2) = [(x-1)(x+2) + (x+1)(x-2)]/[(x-2)(x+2)]$$

$$= (x^2 + x - 2 + x^2 - x - 2)/(x^2 - 4) = (2x^2 - 4)/(x^2 - 4)$$

$$\text{Denominator: } 1 - (x-1)/(x-2) \times (x+1)/(x+2) = 1 - [(x-1)(x+1)]/[(x-2)(x+2)]$$

$$= 1 - (x^2 - 1)/(x^2 - 4) = (x^2 - 4 - x^2 + 1)/(x^2 - 4) = (-3)/(x^2 - 4)$$

$$\text{So, } \tan(A + B) = [(2x^2 - 4)/(x^2 - 4)] / [(-3)/(x^2 - 4)] = (2x^2 - 4)/(-3) = 1$$

$$\Rightarrow 2x^2 - 4 = -3$$

$$\Rightarrow 2x^2 = 1$$

$$\Rightarrow x^2 = 1/2$$

$$\Rightarrow x = \pm\sqrt{1/2} = \pm\sqrt{2}/2$$

$$\text{Answer: } x = \sqrt{2}/2 \text{ or } x = -\sqrt{2}/2$$

(a)(ii) Prove $1 + \sin x/(1 - \sin x) = (\tan x + \sec x)^2$

$$\text{Right side: } (\tan x + \sec x)^2 = \tan^2 x + 2\tan x \sec x + \sec^2 x$$

$$= (\sin x/\cos x)^2 + 2(\sin x/\cos x)(1/\cos x) + (1/\cos x)^2$$

$$= (\sin^2 x + 2\sin x + 1)/\cos^2 x = (1 + \sin x)^2/(1 - \sin^2 x) = (1 + \sin x)^2/((1 - \sin x)(1 + \sin x))$$

$$= (1 + \sin x)/(1 - \sin x)$$

Left side matches right side. Identity proven.

(b) If $2\sin\theta - \cos\theta = 1$, use t-formula to find θ in $0^\circ \leq \theta \leq 180^\circ$

Let $t = \tan(\theta/2)$, then:

$$\sin\theta = 2t/(1 + t^2), \cos\theta = (1 - t^2)/(1 + t^2)$$

$$\text{Substitute: } 2(2t/(1 + t^2)) - (1 - t^2)/(1 + t^2) = 1$$

$$\Rightarrow (4t - (1 - t^2))/(1 + t^2) = 1$$

$$\Rightarrow 4t - 1 + t^2 = 1 + t^2$$

$$\Rightarrow 4t = 2$$

$$\Rightarrow t = 1/2$$

$$\text{Since } t = \tan(\theta/2), \tan(\theta/2) = 1/2$$

$$\theta/2 = \tan^{-1}(1/2) \approx 26.57^\circ, \text{ so } \theta \approx 53.14^\circ$$

$$\text{In } 0^\circ \leq \theta \leq 180^\circ, \theta = 53.14^\circ.$$

$$(c)(i) \text{ Show } \cos\theta + \cos2\theta + \cos3\theta + \cos4\theta = \cos(5\theta/2)\cos(3\theta/2)/\sin(\theta/2)$$

$$\text{Left: } \cos\theta + \cos2\theta + \cos3\theta + \cos4\theta$$

$$\text{Use sum-to-product: } \cos A + \cos B = 2\cos((A+B)/2)\cos((A-B)/2)$$

$$\text{Pair: } (\cos\theta + \cos4\theta) + (\cos2\theta + \cos3\theta)$$

$$\cos\theta + \cos4\theta = 2\cos(5\theta/2)\cos(-3\theta/2) = 2\cos(5\theta/2)\cos(3\theta/2)$$

$$\cos2\theta + \cos3\theta = 2\cos(5\theta/2)\cos(\theta/2)$$

$$\text{Total: } 2\cos(5\theta/2)\cos(3\theta/2) + 2\cos(5\theta/2)\cos(\theta/2) = 2\cos(5\theta/2)[\cos(3\theta/2) + \cos(\theta/2)]$$

$$\cos(3\theta/2) + \cos(\theta/2) = 2\cos(\theta)\cos(\theta/2)$$

$$\text{So, left} = 4\cos(5\theta/2)\cos(\theta)\cos(\theta/2)$$

$$\text{Right: } \cos(5\theta/2)\cos(3\theta/2)/\sin(\theta/2)$$

$$\text{Use } \cos A \cos B = (1/2)[\cos(A+B) + \cos(A-B)]:$$

$$\cos(5\theta/2)\cos(3\theta/2) = (1/2)[\cos(4\theta) + \cos(\theta)]$$

$$\text{Right} = (1/2)[\cos(4\theta) + \cos(\theta)]/\sin(\theta/2)$$

$$\text{Right} = \cos(5\theta/2)\cos(3\theta/2)/\sin(\theta/2)$$

$$(c)(ii) \text{ Verify } \sin(A + B + C) + \sin(A - B - C) - \cos(A + B - C) - \cos(A - B + C) = \tan B \tan C$$

This identity appears incorrect as stated. Test numerically (e.g., $A = 30^\circ$, $B = 45^\circ$, $C = 60^\circ$):

Left $\neq \tan B \tan C$. Likely a typo in the problem. If intended identity is different, please clarify.

$$(d) \text{ Express } 3\sin\theta - 4\cos\theta \text{ in the form } R\sin(\theta - \alpha), \text{ giving } R \text{ and } \alpha$$

$$3\sin\theta - 4\cos\theta = R\sin(\theta - \alpha) = R(\sin\theta \cos\alpha - \cos\theta \sin\alpha)$$

$$\text{Equate: } R \cos\alpha = 3, -R \sin\alpha = -4$$

$$\Rightarrow R \sin \alpha = 4$$

$$R^2 = 3^2 + 4^2 = 25, R = 5$$

$$\cos \alpha = 3/5, \sin \alpha = 4/5$$

$$\alpha = \tan^{-1}(4/3)$$

$$\text{Answer: } 5 \sin(\theta - \tan^{-1}(4/3))$$

6. (a)(i) Probability a keyboard is defective is 0.01. For a sample of 3, construct the probability distribution of defective keyboards

Let X = number of defective keyboards. $X \sim \text{Binomial}(n=3, p=0.01)$.

$$P(X = 0) = C(3,0) \times (0.01)^0 \times (0.99)^3 = 0.970299$$

$$P(X = 1) = C(3,1) \times (0.01)^1 \times (0.99)^2 = 0.029403$$

$$P(X = 2) = C(3,2) \times (0.01)^2 \times (0.99)^1 = 0.000297$$

$$P(X = 3) = C(3,3) \times (0.01)^3 \times (0.99)^0 = 0.000001$$

Distribution:

$$X \mid 0 \quad | \quad 1 \quad | \quad 2 \quad | \quad 3 \quad |$$

$$P(X) \mid 0.970299 \mid 0.029403 \mid 0.000297 \mid 0.000001 \mid$$

(a)(ii) Find the mean and standard deviation (to 2 decimal places)

$$\text{Mean: } E(X) = n \times p = 3 \times 0.01 = 0.03$$

$$\text{Variance: } \text{Var}(X) = n \times p \times (1 - p) = 3 \times 0.01 \times 0.99 = 0.0297$$

$$\text{Standard deviation: } \sqrt{(0.0297)} \approx 0.17$$

$$\text{Answer: Mean} = 0.03, \text{SD} = 0.17$$

(b) Bag R: 5 red, 3 green; Bag P: 7 red, 5 green. Draw 2 from R, 1 from P. Find probability 2 red, 1 green

$$\text{Total ways: } C(8,2) \times C(12,1) = 28 \times 12 = 336$$

$$\text{Favorable: } 2 \text{ red from R} = C(5,2) = 10, 1 \text{ green from P} = 5$$

$$\text{Probability} = (10 \times 5)/336 = 50/336 = 25/168$$

(c)(i) $P(X = k) = k$ if $x = 0$, $2k$ if $x = 1$, $3k$ if $x = 2$, 0 otherwise. Determine k

Sum of probabilities = 1:

$$k + 2k + 3k = 1$$

$$6k = 1$$

$$k = 1/6$$

(c)(ii) Find $P(X < 2)$, $P(X \leq 2)$, $P(X \geq 2)$

$$P(X = 0) = 1/6, P(X = 1) = 2/6 = 1/3, P(X = 2) = 3/6 = 1/2$$

$$P(X < 2) = P(X = 0) + P(X = 1) = 1/6 + 1/3 = 1/2$$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 1$$

$$P(X \geq 2) = P(X = 2) = 1/2$$

(d) If X is discrete, show $E(aX + b) = aE(X) + b$

$$E(aX + b) = \sum (ax + b) \times P(X = x)$$

$$= a \sum x \times P(X = x) + b \sum P(X = x)$$

$$= aE(X) + b \times 1$$

$$= aE(X) + b$$

(e) 6 seeds, probability of germinating 0.9. Probability at most 5 seeds germinate

$$X \sim \text{Binomial}(n=6, p=0.9).$$

$$P(\text{at most 5}) = 1 - P(X = 6)$$

$$P(X = 6) = (0.9)^6 = 0.531441$$

$$P(X \leq 5) = 1 - 0.531441 = 0.468559 \approx 0.47$$

7. (a)(i) If $x(1 - y) \frac{dy}{dx} - 2y = 0$ and $y = 2$ when $x = e$, show that $x^2y e^y = e^2$

$$\text{Rewrite: } x(1 - y) \frac{dy}{dx} - 2y = 0$$

$$\Rightarrow x(1 - y) \frac{dy}{dx} = 2y$$

$$\Rightarrow (1 - y) \frac{dy}{dx} = 2y/x$$

$$\Rightarrow \frac{dy}{dx} = 2y/[x(1 - y)]$$

Separate variables:

$$(1 - y)/(2y) dy = 1/x dx$$

$$\Rightarrow (1/(2y) - 1/2) dy = 1/x dx$$

Integrate:

$$(1/2)\ln|y| - (1/2)y = \ln|x| + c$$

Apply condition $y = 2$ when $x = e$:

$$(1/2)\ln|2| - (1/2)(2) = \ln|e| + c$$

$$\Rightarrow (1/2)\ln 2 - 1 = 1 + c$$

$$\Rightarrow c = (1/2)\ln 2 - 2$$

$$\text{Equation: } (1/2)\ln|y| - (1/2)y = \ln|x| + (1/2)\ln 2 - 2$$

$$\Rightarrow \ln|y| - y = 2\ln|x| + \ln 2 - 4$$

$$\Rightarrow \ln|y| - y = \ln(x^2) + \ln 2 - 4$$

$$\Rightarrow \ln(y/x^2) - y = \ln 2 - 4$$

$$\Rightarrow y/x^2 = e^{(y + \ln 2 - 4)} = (2/e^4) e^y$$

$$\Rightarrow x^2 y e^y = e^{4/2} \times 2 = e^4$$

$$\text{Show: } x^2 y e^y = e^2$$

From above, $x^2 y e^y = e^4$, which does not match e^2 . Likely a typo in the problem (should be e^4).

(a)(ii) Solve $2x - y \, dy/dx = 2x + y + 2$ given $y = 1$ when $x = 2$

$$\text{Rewrite: } 2x - y \, dy/dx = 2x + y + 2$$

$$\Rightarrow y \, dy/dx = -2$$

Integrate:

$$y^2/2 = -2x + c$$

$$\Rightarrow y^2 = -4x + c$$

At $x = 2$, $y = 1$:

$$1 = -4(2) + c$$

$$\Rightarrow c = 9$$

$$\Rightarrow y^2 = -4x + 9$$

$$\Rightarrow y = \pm\sqrt{(-4x + 9)}$$

Since $y = 1$ at $x = 2$, $y = \sqrt{-4x + 9}$ (positive branch).

(b) Form a differential equation whose solution is $y = A e^{2x} + B e^{-x}$ where A and B are arbitrary constants

$$y = A e^{2x} + B e^{-x}$$

$$dy/dx = 2A e^{2x} - B e^{-x}$$

$$d^2y/dx^2 = 4A e^{2x} + B e^{-x}$$

Express in terms of y :

$$d^2y/dx^2 = 4A e^{2x} + B e^{-x}$$

$$y = A e^{2x} + B e^{-x}$$

$$dy/dx - 2y = (2A e^{2x} - B e^{-x}) - 2(A e^{2x} + B e^{-x}) = -3B e^{-x}$$

$$d^2y/dx^2 - 2 dy/dx = 4A e^{2x} + B e^{-x} - 2(2A e^{2x} - B e^{-x}) = -2y$$

$$\text{Differential equation: } d^2y/dx^2 - 2 dy/dx + 2y = 0$$

(c)(i) Tank with 1000 liters, 10 kg salt. Poured off at 20 liters/min, pure water added at 20 liters/min. Find mass of salt after 5 minutes

Let $S(t)$ be salt mass at time t (in kg). Volume remains 1000 liters.

$$\text{Salt concentration} = S/1000 \text{ kg/liter}$$

$$\text{Salt poured off: } (S/1000) \times 20 = S/50 \text{ kg/min}$$

$$dS/dt = -S/50$$

$$dS/S = -dt/50$$

$$\ln|S| = -t/50 + c$$

$$S = c e^{-(t/50)}$$

$$\text{At } t = 0, S = 10:$$

$$10 = c$$

$$S = 10 e^{-(t/50)}$$

$$\text{At } t = 5:$$

$$S = 10 e^{(-5/50)} = 10 e^{(-0.1)} \approx 9.05 \text{ kg}$$

(c)(ii) How long until mass of salt falls to 5 kg?

$$5 = 10 e^{(-t/50)}$$

$$e^{(-t/50)} = 1/2$$

$$-t/50 = \ln(1/2)$$

$$t = 50 \ln(2) \approx 34.66 \text{ minutes}$$

(d) Find the general solution of $dy/dx - 4 dy/dy + 3y = 10 e^{3x}$

$$\text{Corrected: } d^2y/dx^2 - 4 dy/dx + 3y = 10 e^{3x}$$

$$\text{Homogeneous: } d^2y/dx^2 - 4 dy/dx + 3y = 0$$

$$\text{Characteristic: } r^2 - 4r + 3 = 0$$

$$\Rightarrow (r - 1)(r - 3) = 0$$

$$\Rightarrow r = 1, 3$$

$$y_h = c_1 e^x + c_2 e^{3x}$$

$$\text{Particular: } y_p = A e^{3x}$$

$$y_p' = 3A e^{3x}, y_p'' = 9A e^{3x}$$

$$9A e^{3x} - 4(3A e^{3x}) + 3(A e^{3x}) = 10 e^{3x}$$

$$y_p = Ax e^{3x}:$$

$$y_p' = A e^{3x} + 3Ax e^{3x}, y_p'' = 6A e^{3x} + 9Ax e^{3x}$$

$$(6A + 9Ax - 12A - 12Ax + 3Ax) e^{3x} = 10 e^{3x}$$

$$(6A - 3Ax) e^{3x} = 10 e^{3x}$$

$$6A = 10, A = 5/3$$

$$\text{General solution: } y = c_1 e^x + c_2 e^{3x} + (5/3)x e^{3x}$$

8. (a)(i) If $9y^2 - 54y - 25x^2 + 200x - 544 = 0$ is the hyperbola equation, find center, vertices, foci, equation of asymptotes

$$\text{Rewrite: } 9(y^2 - 6y) - 25(x^2 - 8x) - 544 = 0$$

$$9[(y - 3)^2 - 9] - 25[(x - 4)^2 - 16] = 544$$

$$9(y - 3)^2 - 25(x - 4)^2 = 225$$

$$(y - 3)^2/25 - (x - 4)^2/9 = 1$$

Center: (4, 3)

$$a = 5, b = 3$$

$$\text{Vertices: } (4, 3 \pm 5) = (4, 8), (4, -2)$$

$$c = \sqrt{a^2 + b^2} = \sqrt{25 + 9} = \sqrt{34}$$

$$\text{Foci: } (4, 3 \pm \sqrt{34})$$

$$\text{Asymptotes: } (y - 3) = \pm(5/3)(x - 4)$$

(a)(ii) Convert $x + y = 4x$ into polar equation

$$x = r \cos\theta, y = r \sin\theta$$

$$r \cos\theta + r \sin\theta = 4(r \cos\theta)$$

$$r(\cos\theta + \sin\theta - 4\cos\theta) = 0$$

$$r = 0 \text{ or } \cos\theta + \sin\theta - 4\cos\theta = 0$$

$$\Rightarrow r = 0 \text{ or } r = 3\cos\theta/(\cos\theta + \sin\theta)$$

$$\text{Polar equation: } r = 3\cos\theta/(\cos\theta + \sin\theta)$$

(a)(iii) Convert [1, -1] into polar coordinates

Cartesian (1, -1):

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}(-1/1) = -\pi/4$$

$$\text{Polar: } (\sqrt{2}, -\pi/4)$$

(b) Find equation of tangent and normal at $P(a\cos\alpha, b\sin\alpha)$ to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$

$$\text{Ellipse: } x^2/a^2 + y^2/b^2 = 1$$

Point $P(a\cos\alpha, b\sin\alpha)$.

Differentiate: $2x/a^2 + 2y/b^2 \, dy/dx = 0$

$$dy/dx = -(b^2x)/(a^2y)$$

$$\text{At } P: dy/dx = -(b^2 \cos\alpha)/(a^2 \sin\alpha) = -(b/a)\cot\alpha$$

$$\text{Tangent: } y - b\sin\alpha = -(b/a)\cot\alpha (x - a\cos\alpha)$$

$$\Rightarrow bx \cot\alpha + ay = ab(\cot\alpha \cos\alpha + \sin\alpha)$$

Normal slope: $(a/b)\tan\alpha$

$$y - b\sin\alpha = (a/b)\tan\alpha (x - a\cos\alpha)$$

(c)(i) Define a conic section

A conic section is the curve obtained by intersecting a plane with a double cone. It can be a circle, ellipse, parabola, or hyperbola, depending on the angle of intersection.

(c)(ii) Man running a race-course, sum of distances from two flag posts to him is always 10 meters. Sum of distance between flag posts is 8 meters. Find equation of path traced by the man

This is an ellipse: sum of distances to foci $= 2a = 10$, $a = 5$.

Distance between foci $= 2ae = 8$, $ae = 4$

$$e = 4/5$$

$$b^2 = a^2(1 - e^2) = 25(1 - (4/5)^2) = 25(1 - 16/25) = 9$$

Foci at $(\pm 4, 0)$, center at $(0, 0)$.

$$\text{Equation: } x^2/25 + y^2/9 = 1$$

(d) Sketch the graph of $r = 2(1 + \sin\theta)$

This is a cardioid.

At $\theta = 0$, $r = 2$; $\theta = \pi/2$, $r = 4$; $\theta = \pi$, $r = 2$; $\theta = 3\pi/2$, $r = 0$.

Graph: Heart-shaped curve, cusp at $(0, 0)$, extends to $(0, 4)$.

