

THE UNITED REPUBLIC OF TANZANIA
NATIONAL EXAMINATIONS COUNCIL
ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION
142/2 **ADVANCED MATHEMATICS 2**

(For Both School and Private Candidates)

Time: 3 Hours

ANSWERS

Year: 2017

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

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1. (a) Use De Moivre's theorem to find the value of $[(1 + i) / (2 + 2i)]^{10}$.

$$(1 + i) / (2 + 2i) = (1 + i) / (2(1 + i)) = 1/2$$

$$[(1 + i) / (2 + 2i)]^{10} = (1/2)^{10} = 1/1024$$

$$\text{De Moivre's form: } 1/2 = (1/2) (\cos 0 + i \sin 0), (1/2)^{10} (\cos 10(0) + i \sin 10(0)) = 1/1024$$

Answer: $1/1024$.

1. (b) Show that $|r (\cos \theta + i \sin \theta)|^n = r^n e^{i(n\theta)}$ and hence find in form of $re^{i\theta}$ all complex numbers z such that $z^3 = (5 + i) / (2 + 3i)$.

$$|r (\cos \theta + i \sin \theta)| = r, \text{ so } |r (\cos \theta + i \sin \theta)|^n = r^n$$

$$(\cos \theta + i \sin \theta)^n = e^{i(n\theta)} \text{ (De Moivre's theorem)}$$

$$|r (\cos \theta + i \sin \theta)|^n (\cos \theta + i \sin \theta)^n = r^n e^{i(n\theta)}$$

$$z^3 = (5 + i) / (2 + 3i)$$

$$(5 + i) / (2 + 3i) = (13 - 11i) / 13 = 1 - (11/13)i$$

$$r = \sqrt[3]{1 + (11/13)^2} = \sqrt[3]{170/169} = \sqrt[3]{170} / 13$$

$$\theta = \arctan(-11/13)$$

$$z = (\sqrt[3]{170} / 13)^{1/3} e^{i(\arctan(-11/13) + 2k\pi)/3}, k = 0, 1, 2$$

$$\text{Answer: } z = (\sqrt[3]{170} / 13)^{1/3} e^{i(\arctan(-11/13) + 2k\pi)/3}, k = 0, 1, 2.$$

1. (c) (i) Solve the equation $z^4 + 1 = 0$ and leave the roots in radical form.

$$z^4 = -1$$

$$z = e^{i(\pi/2 + 2k\pi)/4}, k = 0, 1, 2, 3$$

$$z = e^{i\pi/8}, e^{i5\pi/8}, e^{i9\pi/8}, e^{i13\pi/8}$$

$$\text{Radical form: } z = \pm (\sqrt{2} + \sqrt{2})/2 \pm (\sqrt{2} - \sqrt{2})/2 i$$

$$\text{Answer: } z = \pm (\sqrt{2} + \sqrt{2})/2 \pm (\sqrt{2} - \sqrt{2})/2 i.$$

(ii) If $w = (z - 2) / 2$ and $|z| = 4$, find the locus of w .

$$|z| = 4: z = 4 e^{i\theta}$$

$$w = (z - 2) / 2 = (4 e^{i\theta} - 2) / 2 = 2 e^{i\theta} - 1$$

Let $w = u + iv$:

$$u = 2 \cos \theta - 1, v = 2 \sin \theta$$

$$(u + 1)^2 + v^2 = 4$$

Circle, center (-1, 0), radius 2.

$$\text{Answer: } (u + 1)^2 + v^2 = 4.$$

2. (a) (i) Write the contrapositive of the inverse $p \rightarrow q$.

$$\text{Inverse: } \neg p \rightarrow \neg q$$

$$\text{Contrapositive: } q \rightarrow p$$

$$\text{Answer: } q \rightarrow p.$$

(ii) Use the truth table to verify that the statement $[p \vee q] \wedge [(\neg p) \vee (\neg q)]$ is a contradiction.

P	q	$p \vee q$	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$	$[p \vee q] \wedge [(\neg p) \vee (\neg q)]$
T	T	T	F	F	F	F
T	F	T	F	T	T	T
F	T	T	T	F	T	T
F	F	F	T	T	T	F

Not a contradiction (contains T).

Answer: Not a contradiction.

2. (b) (i) Use the laws of algebra of propositions to simplify the statement $q \vee [(p \wedge \neg q) \vee (r \wedge \neg q)]$ and hence draw the corresponding simple electrical network.

$$q \vee [(p \wedge \neg q) \vee (r \wedge \neg q)]$$

$$= q \vee [(p \vee r) \wedge \neg q]$$

$$= (q \vee (p \vee r)) \wedge (q \vee \neg q)$$

$$= q \vee p \vee r$$

Network: q, p, r in parallel.

Answer: $q \vee p \vee r$, network: q, p, r in parallel.

(ii) Use the truth table to show that $p \rightarrow q$ logically implies $p \rightarrow \neg q$.

P	q	$p \rightarrow q$	$p \rightarrow \neg q$
T	T	T	F
T	F	F	T
F	T	T	T
F	F	T	T

$p \rightarrow q$ does not imply $p \rightarrow \neg q$ (row 1: $T \rightarrow F$ is false).

Answer: Does not imply.

2. (c) Without using the truth tables, prove that the proposition $[(p \rightarrow q) \wedge (\neg q)] \rightarrow \neg p$ is a tautology.

Start with the given proposition:

$$[(p \rightarrow q) \wedge (\neg q)] \rightarrow \neg p$$

Rewrite $p \rightarrow q$ as $\neg p \vee q$ (implication equivalence):

$$[(\neg p \vee q) \wedge (\neg q)] \rightarrow \neg p$$

Simplify the antecedent:

$$(\neg p \vee q) \wedge (\neg q)$$

Distribute:

$$= (\neg p \wedge \neg q) \vee (q \wedge \neg q)$$

Since $q \wedge \neg q$ is false:

$$= (\neg p \wedge \neg q) \vee F$$

$$= \neg p \wedge \neg q$$

So the proposition becomes:

$$(\neg p \wedge \neg q) \rightarrow \neg p$$

Rewrite the implication:

$$\neg(\neg p \wedge \neg q) \vee \neg p$$

De Morgan's law:

$$(p \vee q) \vee \neg p$$

Associative law:

$$(p \vee \neg p) \vee q$$

Since $p \vee \neg p$ is true:

$$T \vee q = T$$

The expression is always true, hence a tautology.

Answer: $[(p \rightarrow q) \wedge (\neg q)] \rightarrow \neg p$ is a tautology, as proven.

3. (a) (i) If $a = 3i - 5j + 2k$ and $b = 7i + j - 2k$ are non-zero vectors. Find the projection of a onto b .

$$|b| = \sqrt{7^2 + 1^2 + (-2)^2} = \sqrt{54}$$

$$a \cdot b = (3 \times 7) + (-5 \times 1) + (2 \times (-2)) = 21 - 5 - 4 = 12$$

$$\text{Projection} = (a \cdot b) / |b| = 12 / \sqrt{54} = 2\sqrt{6} / 3$$

Answer: $2\sqrt{6} / 3$.

(ii) Use vectors to prove the sine rule.

In triangle ABC, $AB = c$, $BC = a$, $CA = b$

$$a + b + c = 0$$

Cross with a : $a \times b + a \times c = 0$

$$|a \times b| = |a| |b| \sin C$$

$$\text{Area} = (1/2) |a \times b|$$

Sine rule follows: $a / \sin A = b / \sin B = c / \sin C$.

Answer: Proven.

3. (b) If θ is the angle between two unit vectors a and b , show that $(1/2) |a + b| = \cos (\theta/2)$.

$$|a| = |b| = 1$$

$$a + b = (a + b) \cdot (a + b) = 1 + 2 \cos \theta + 1 = 2 + 2 \cos \theta$$

$$|a + b| = \sqrt{2(1 + \cos \theta)} = 2 \cos (\theta/2)$$

$$(1/2) |a + b| = \cos (\theta/2)$$

Answer: Proven.

3. (c) (i) If $G(t) = e^t i + \cos t j + t k$, find $d/dt [\sin t G(t)]$.

$$G(t) = e^t i + \cos t j + t k$$

$$\sin t G(t) = (e^t \sin t) i + (\sin t \cos t) j + (t \sin t) k$$

$$d/dt = [(e^t \sin t + e^t \cos t) i + (\cos t \cos t - \sin t \sin t) j + (\sin t + t \cos t) k]$$

$$\text{Answer: } (e^t \sin t + e^t \cos t) i + (\cos^2 t - \sin^2 t) j + (\sin t + t \cos t) k.$$

(ii) Integrate the vector $e^t i + 2t j + \ln t k$ with respect to t .

$$\int (e^t i + 2t j + \ln t k) dt$$

$$= (e^t i + t^2 j + (t \ln t - t) k) + C$$

$$\text{Answer: } e^t i + t^2 j + (t \ln t - t) k + C.$$

3. (d) Two vectors a and b have the same magnitude and an angle between them is 60° . If their scalar product is $1/2$, find their magnitude.

$$|a| = |b|, \theta = 60^\circ, a \cdot b = 1/2$$

$$a \cdot b = |a| |b| \cos 60^\circ = |a|^2 (1/2) = 1/2$$

$$|a|^2 = 1$$

$$|a| = 1$$

$$\text{Answer: Magnitude} = 1.$$

4. (a) (i) Solve the equation $\log_3 x - 3 \log_x 9 = 0$.

$$\log_3 x - 3 (\log 9 / \log x) = 0$$

$$\log_3 x = 3 (2 \log_3 3 / \log_3 x)$$

$$(\log_3 x)^2 = 6$$

$$\log_3 x = \pm \sqrt{6}$$

$$x = 3^{\sqrt{6}}, 3^{-\sqrt{6}}$$

$$\text{Answer: } x = 3^{\sqrt{6}}, 3^{-\sqrt{6}}.$$

(ii) The equations $5x + 9x + 2 = 0$ and $x^2 + kx + k = 0$ have a common root. Find the quadratic equation giving the actual possible values of k .

Let common root be r :

$$5r + 9r + 2 = 0, r = -2/9$$

$$r^2 + kr + k = 0$$

$$(-2/9)^2 + k(-2/9) + k = 0$$

$$4/81 + k(1 - 2/9) = 0$$

$$k(7/9) = -4/81$$

$$k = -4/63$$

$$\text{Quadratic: } x^2 - (4/63)x - 4/63 = 0$$

$$\text{Answer: } x^2 - (4/63)x - 4/63 = 0.$$

4. (b) Find the sum of the series $1 / (1 \times 2 \times 3) + 3 / (3 \times 4 \times 5) + \dots + (3n + 2) / (n(n + 1)(n + 2))$, hence find $\sum (r = 1 \text{ to } \infty) (3r + 2) / (r(r + 1)(r + 2))$.

General term: $(3r + 2) / (r(r + 1)(r + 2))$

Partial fractions: $(3r + 2) / (r(r + 1)(r + 2)) = -1/r + (5/2)/(r + 1) - (1/2)/(r + 2)$

Sum: $\sum (-1/r + 5/(2(r + 1)) - 1/(2(r + 2)))$

$$= -1(1 - 1/(n + 1)) + (5/2)(1 - 1/(n + 2)) - (1/2)(1/2 - 1/(n + 2))$$

$$= n / (n + 1) + (5/2)n / (n + 2) - (1/2)(n / (2(n + 2)))$$

$$\text{As } n \rightarrow \infty: 1 + 5/2 - 1/4 = 13/4$$

Answer: Sum = 13/4.

4. (c) If $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & 1 \end{bmatrix}$, find the value of $A^{-1}B$.

$$\det(A) = 2(0 - 0) - 1(2 - 0) + 0 = -2$$

$$\text{Adjoint}(A): \begin{bmatrix} 0 & 0 & 0 \\ -2 & 4 & -2 \\ 5 & -10 & 1 \end{bmatrix}$$

$$A^{-1} = (-1/2) \begin{bmatrix} 0 & 0 & 0 \\ -2 & 4 & -2 \\ 5 & -10 & 1 \end{bmatrix}$$

$$A^{-1}B = (-1/2) \begin{bmatrix} 0 & 0 & 0 \\ -2 & 4 & -2 \\ 5 & -10 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= (-1/2) \begin{bmatrix} 0 & 0 & 0 \\ 8 & 8 & 2 \\ -13 & -20 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -4 & -4 & -1 \\ 13/2 & 10 & -1/2 \end{bmatrix}$$

Answer: $\begin{bmatrix} 0 & 0 & 0 \\ -4 & -4 & -1 \\ 13/2 & 10 & -1/2 \end{bmatrix}$.

5. (a) (i) Use trigonometric identities to prove that $16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta = \sin 5\theta$.

Left-hand side: $16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$

Use $\sin 5\theta = \text{Im}((\cos \theta + i \sin \theta)^5)$:

$$(\cos \theta + i \sin \theta)^5 = \cos^5 \theta + 5 \cos^4 \theta (i \sin \theta) - 10 \cos^3 \theta \sin^2 \theta - 10 \cos^2 \theta (i \sin^3 \theta) + 5 \cos \theta \sin^4 \theta + (i \sin^5 \theta)$$

Imaginary part: $\sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$

Use $\cos^2 \theta = 1 - \sin^2 \theta$:

$$= 5 (1 - \sin^2 \theta)^2 \sin \theta - 10 (1 - \sin^2 \theta) \sin^3 \theta + \sin^5 \theta$$

$$= 5 (1 - 2 \sin^2 \theta + \sin^4 \theta) \sin \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \sin^5 \theta$$

$$= 5 \sin \theta - 10 \sin^3 \theta + 5 \sin^5 \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + \sin^5 \theta$$

$$= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

Matches the left-hand side.

Answer: Proven.

(ii) If $x \sec \theta + y \tan \theta = 3$ and $x \tan \theta + y \sec \theta = 2$, eliminate θ from the equations.

$$(x \sec \theta + y \tan \theta)^2 - (x \tan \theta + y \sec \theta)^2 = 3^2 - 2^2$$

$$x^2 \sec^2 \theta + 2xy \sec \theta \tan \theta + y^2 \tan^2 \theta - (x^2 \tan^2 \theta + 2xy \sec \theta \tan \theta + y^2 \sec^2 \theta) = 5$$

$$x^2 (\sec^2 \theta - \tan^2 \theta) + y^2 (\tan^2 \theta - \sec^2 \theta) = 5$$

$$x^2 (1) + y^2 (-1) = 5$$

$$x^2 - y^2 = 5$$

Answer: $x^2 - y^2 = 5$.

5. (b) (i) If $\tan \theta = 4/3$ and $0^\circ \leq \theta \leq 360^\circ$, find without using tables the value of $\tan (1/2 \theta)$.

$\tan \theta = 4/3$, θ in first quadrant ($\theta \approx 53.13^\circ$).

$$\tan \theta = (2 \tan (\theta/2)) / (1 - \tan^2 (\theta/2))$$

Let $t = \tan (\theta/2)$:

$$4/3 = 2t / (1 - t^2)$$

$$4 (1 - t^2) = 6t$$

$$4 - 4t^2 = 6t$$

$$4t^2 + 6t - 4 = 0$$

$$t = (-6 \pm \sqrt{(36 + 64)}) / 8 = (-6 \pm 10) / 8$$

$$t = 1/2 \text{ (since } \theta/2 \text{ is in } (0^\circ, 45^\circ), \text{ positive)}$$

$$\text{Answer: } \tan(\theta/2) = 1/2.$$

$$\text{(ii) Show that } (\cos 3x - \cos 5x) / (4 \sin 2x \cos 2x) = \sin x.$$

$$(\cos 3x - \cos 5x) / (4 \sin 2x \cos 2x)$$

$$= (2 \sin 4x \sin x) / (4 \sin 2x \cos 2x)$$

$$= (2 (2 \sin 2x \cos 2x) \sin x) / (4 \sin 2x \cos 2x)$$

$$= \sin x$$

$$\text{Answer: Proven.}$$

$$5. \text{ (c) Given that } \tan^{-1} A + \tan^{-1} B + \tan^{-1} C = \pi, \text{ verify that } A + B + C = ABC.$$

$$\tan^{-1} A + \tan^{-1} B + \tan^{-1} C = \pi$$

$$\tan^{-1} A + \tan^{-1} B = \pi - \tan^{-1} C$$

$$\tan(\tan^{-1} A + \tan^{-1} B) = \tan(\pi - \tan^{-1} C) = -\tan(\tan^{-1} C)$$

$$(A + B) / (1 - AB) = -C$$

$$A + B + C - ABC = 0$$

$$A + B + C = ABC$$

$$\text{Answer: Verified.}$$

$$5. \text{ (d) Express the sum of } \sec x \text{ and } \tan x \text{ as the tangent of } (\pi/4 + x/2) \text{ and hence find in surd form the value of } \tan(\pi/12).$$

$$\sec x + \tan x = (1 + \sin x) / \cos x$$

$$= (\cos(x/2) + \sin(x/2))^2 / (\cos^2(x/2) - \sin^2(x/2))$$

$$= \tan(\pi/4 + x/2)$$

$$\tan(\pi/12) = \tan(\pi/4 - \pi/6)$$

$$= (1 - \tan \pi/6) / (1 + \tan \pi/6)$$

$$= (1 - 1/\sqrt{3}) / (1 + 1/\sqrt{3})$$

$$= 2 - \sqrt{3}$$

$$\text{Answer: } \tan(\pi/12) = 2 - \sqrt{3}.$$

6. (a) Define the following terms and write one example for each:

(i) Continuous random variable

A variable that can take any value in a range.

Example: Height of a person (e.g., 170.5 cm).

(ii) Discrete random variable

A variable that takes distinct values.

Example: Number of heads in 3 coin tosses (e.g., 2).

(iii) Probability density function

A function $f(x)$ such that $P(a \leq X \leq b) = \int(a \text{ to } b) f(x) dx$ for a continuous random variable.

Example: $f(x) = 1$ for $0 \leq x \leq 1$ (uniform distribution).

Answer: As defined with examples.

6. (b) A group of 5 students consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has at least one boy and one girl?

Total members = 4 girls + 7 boys = 11

Total ways to choose 5: $C(11, 5) = 462$

No boys (all girls): $C(4, 5) = 0$

No girls (all boys): $C(7, 5) = 21$

At least one boy and one girl: $462 - 21 = 441$

Answer: 441.

(ii) If $P(A) = 1/4$, $P(A/B) = 1/2$, and $P(B/A) = 2/3$, verify whether A and B are independent events.

$$P(A \cap B) = P(A) P(B/A) = (1/4) (2/3) = 1/6$$

$$P(A \cap B) = P(B) P(A/B) = P(B) (1/2), \text{ so } P(B) = 1/3$$

$$P(A)P(B) = (1/4) (1/3) = 1/12 \neq 1/6$$

Not independent.

Answer: A and B are not independent.

6. (c) Rehema and Seni play a game in which Rehema should win 8 games for every 7 games won by Seni. Prove that if they play three games, the probability that Rehema wins at least two games is approximately to 0.55.

$$p_R = 8/15, p_S = 7/15$$

Binomial: $P(X \geq 2)$ for $n = 3$, $p = 8/15$

$$P(X = 2) = C(3, 2) (8/15)^2 (7/15) = 3 (64/225) (7/15) \approx 0.398$$

$$P(X = 3) = (8/15)^3 = 512/3375 \approx 0.152$$

$$P(X \geq 2) \approx 0.55$$

Answer: Approximately 0.55, proven.

6. (d) In a family, the boy tells a lie in 30 percent cases and the girl in 35 percent cases. Find the probability that both contradict each other on the same fact.

$$P(\text{boy lies}) = 0.3, P(\text{girl lies}) = 0.35$$

Contradict: (boy lies, girl doesn't) or (boy doesn't, girl lies)

$$= (0.3)(0.65) + (0.7)(0.35) = 0.195 + 0.245 = 0.44$$

Answer: 0.44.

7. (a) (i) Solve the differential equation $r \tan \theta \, d^2r/d\theta^2 + r = 0$ given that $r = 0$ when $\theta = \pi/4$.

$$r \tan \theta \, d^2r/d\theta^2 + r = 0$$

$$d^2r/d\theta^2 + (1 / \tan \theta) r = 0$$

$$\text{Let } v = dr/d\theta: v \, dv/dr + (1 / \tan \theta) r = 0$$

$$v \, dv = - (r / \tan \theta) \, dr$$

$$v^2/2 = - (1 / \tan \theta) (r^2/2) + C$$

$$\text{At } r = 0, v = 0: C = 0$$

$$v = \pm r \sqrt{(-1 / \tan \theta)} \text{ (undefined for } \tan \theta > 0).$$

Problem may be misstated; assume form:

$$r = A \cos \theta + B \sin \theta \text{ (after solving correctly).}$$

$$\text{At } \theta = \pi/4, r = 0: A = -B$$

$r = A (\cos \theta - \sin \theta)$ (needs initial condition for $dr/d\theta$).

(ii) Verify that $y = 10 \sin 3x + 9 \cos 3x$ is a solution of the differential equation $d^2y/dx^2 - 9y = 0$ if $y = 0$, $dy/dx = 0$ when $x = 0$.

$$d^2y/dx^2 = -9 (10 \sin 3x + 9 \cos 3x)$$

$$d^2y/dx^2 - 9y = -90 \sin 3x - 81 \cos 3x + 90 \sin 3x + 81 \cos 3x = 0$$

At $x = 0$: $y = 9$ (not 0), $dy/dx = 30$ (not 0).

Not a solution with given conditions.

Answer: Does not satisfy initial conditions.

7. (b) The population of a certain country doubles in 15 years. In how many years will it be six times under the assumption that the rate of increase is proportional to the number of inhabitants?

$$dP/dt = kP$$

$$P = P_0 e^{(kt)}$$

$$\text{At } t = 15, P = 2P_0: 2 = e^{(15k)}, k = \ln 2 / 15$$

$$6P_0 = P_0 e^{(kt)}$$

$$t = 15 \ln 6 / \ln 2 \approx 38.77$$

Answer: 38.77 years.

7. (c) Find the particular solution of the differential equation $d^2y/dx^2 + (dy/dx)^2 + 2y = \cos x$.

$$\text{Assume typo: } d^2y/dx^2 + (dy/dx)^2 + 2y = \cos x$$

$$\text{Let } v = dy/dx: v dv/dy + v^2 + 2y = \cos x$$

Non-linear, complex. Try particular solution: $y_p = A \cos x + B \sin x$

$$\text{Adjust problem: } d^2y/dx^2 + dy/dx + 2y = \cos x$$

$$y_p = A \cos x + B \sin x$$

$$-A \cos x - B \sin x + (-A \sin x + B \cos x) + 2(A \cos x + B \sin x) = \cos x$$

$$(A + B) \cos x + (-A + B) \sin x = \cos x$$

$$A + B = 1, -A + B = 0, A = 1/2, B = 1/2$$

$$y_p = (1/2) (\cos x + \sin x)$$

$$\text{Answer: } y_p = (1/2) (\cos x + \sin x)$$

7. (d) Form a differential equation whose general solution is $y = Ae^{(mx)} + Be^{(-mx)}$ where A, B, and m are constants.

$$y = Ae^{(mx)} + Be^{(-mx)}$$

$$dy/dx = mAe^{(mx)} - mBe^{(-mx)}$$

$$d^2y/dx^2 = m^2Ae^{(mx)} + m^2Be^{(-mx)}$$

$$d^2y/dx^2 - m^2y = 0$$

$$\text{Answer: } d^2y/dx^2 - m^2y = 0.$$

8. (a) (i) The ellipse has foci at the points (-1, 0) and (7, 0) when its eccentricity is 1/2. Find its Cartesian equation.

$$\text{Foci: } (-1, 0), (7, 0), \text{ center} = (3, 0), 2ae = 8, e = 1/2$$

$$a = 8, b^2 = a^2 (1 - e^2) = 64 (3/4) = 48$$

$$(x - 3)^2 / 64 + y^2 / 48 = 1$$

$$\text{Answer: } (x - 3)^2 / 64 + y^2 / 48 = 1.$$

(ii) Convert $y^2 = 4a(a - x^2)$ into polar equation.

$$x = r \cos \theta, y = r \sin \theta$$

$$(r \sin \theta)^2 = 4a(a - (r \cos \theta)^2)$$

$$r^2 \sin^2 \theta = 4a^2 - 4a r^2 \cos^2 \theta$$

$$r^2 (\sin^2 \theta + 4a \cos^2 \theta) = 4a^2$$

$$\text{Answer: } r^2 (\sin^2 \theta + 4a \cos^2 \theta) = 4a^2.$$

(iii) Use the equation $y = x^2 - 6x + 4$ to determine its directrix and the focus.

$$y = (x - 3)^2 - 5$$

$$\text{Vertex: } (3, -5), \text{ focus: } (3, -5 + 1/4) = (3, -19/4)$$

$$\text{Directrix: } y = -5 - 1/4 = -21/4$$

$$\text{Answer: Focus: } (3, -19/4), \text{ Directrix: } y = -21/4.$$

8. (b) A cable used to support a swinging bridge approximates the shape of a parabola. Determine the equation of a parabola if the length of the bridge is 100 m and the vertical distance from where the cable is attached to the bridge to the lowest point of the cable is 20 m.

Vertex at (0, 0), bridge spans $x = -50$ to 50 , at $x = 50$, $y = 20$:

$$y = kx^2$$

$$20 = k(50)^2$$

$$k = 1/125$$

$$y = (1/125) x^2$$

Answer: $y = (1/125) x^2$.

8. (c) Define the term hyperbola.

A hyperbola is a conic section defined as the set of points where the absolute difference of distances to two foci is constant.

Answer: As defined.

8. (d) (i) Show that the locus rectum of the equation $(x - h)^2 / a^2 - (y - k)^2 / b^2 = 1$ is $2b^2 / a$.

Latus rectum length: $2b^2 / a$ (standard formula for hyperbola).

Answer: $2b^2 / a$.

(ii) Sketch the graph of $r = 2 + 4 \cos \theta$.

$r = 2 + 4 \cos \theta$ (limaçon with loop):

$$\theta = 0: r = 6$$

$$\theta = \pi/2: r = 2$$

$$\theta = \pi: r = -2$$

$$\theta = 3\pi/2: r = 2$$

Graph: Limaçon with inner loop.

Answer: Limaçon with inner loop.