

THE UNITED REPUBLIC OF TANZANIA
NATIONAL EXAMINATIONS COUNCIL
ADVANCED CERTIFICATE OF SECONDARY EDUCATION EXAMINATION
142/2 **ADVANCED MATHEMATICS 2**

(For Both School and Private Candidates)

Time: 3 Hours

ANSWERS

Year: 2022

Instructions

1. This paper consists of section A and B.
2. Answer all questions in section A and two questions from section B.
3. **All** work done and answers of each question must be shown clearly.
4. NECTA'S Mathematical tables and Non-programmable calculations may be used
5. All writing must be in **black** or **blue** ink, **except** drawing which must be in pencil.

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Prepared by: Maria Marco for TETE

1. (a) The time taken by John to deliver milk to the High Street is normally distributed with mean 12 minutes and standard deviation 2 minutes. If he delivers milk every day, estimate the number of days during the year when he takes longer than 17 minutes (1 year = 365 days).

Mean $\mu = 12$, SD $\sigma = 2$.

$$P(X > 17) = P(Z > (17 - 12) / 2) = P(Z > 2.5)$$

$$P(Z > 2.5) \approx 1 - 0.9938 = 0.0062$$

$$\text{Number of days} = 365 \times 0.0062 \approx 2.263$$

Answer: Approximately 2 days.

1. (b) Suppose that a group of people in a village attending hospital has been categorized according to the incidence of two diseases:

Sex	Malaria	Typhoid
Male	16	12
Female	12	10

Find the probability that the person chosen is a female given the person is suffering from malaria.

$$\text{Total malaria} = 16 + 12 = 28$$

$$\text{Female with malaria} = 12$$

$$P(\text{Female} | \text{Malaria}) = 12 / 28 = 3 / 7$$

Answer: The probability is 3/7.

1. (c) In how many ways can a hand of 4 cards be chosen from an ordinary pack of 52 playing cards?

$$\text{Number of ways} = C(52, 4)$$

$$= (52 \times 51 \times 50 \times 49) / (4 \times 3 \times 2 \times 1)$$

$$= 6497400 / 24 = 270725$$

Answer: 270725 ways.

2. (a) Write the converse and inverse of the statement “If you score an A grade in a logic test, then I will buy you a new car” in words and symbolic form.

Statement: $p \rightarrow q$ (p: score an A, q: buy a new car)

Converse: $q \rightarrow p$

"If I buy you a new car, then you scored an A grade in a logic test."

Inverse: $\neg p \rightarrow \neg q$

"If you do not score an A grade in a logic test, then I will not buy you a new car."

Answer: Converse: "If I buy you a new car, then you scored an A grade in a logic test" ($q \rightarrow p$).

Inverse: "If you do not score an A grade in a logic test, then I will not buy you a new car" ($\neg p \rightarrow \neg q$).

2. (b) Using a truth table, examine whether $[\neg(p \rightarrow (p \wedge q))] \vee (p \rightarrow q)$ is equivalent to $(q \rightarrow p) \wedge (p \rightarrow q)$.

Truth table:

$p \mid q \mid p \wedge q \mid p \rightarrow (p \wedge q) \mid \neg(p \rightarrow (p \wedge q)) \mid p \rightarrow q \mid [\neg(p \rightarrow (p \wedge q))] \vee (p \rightarrow q) \mid q \rightarrow p \mid (q \rightarrow p) \wedge (p \rightarrow q)$

0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1

0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0

1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0

1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1

Columns $[\neg(p \rightarrow (p \wedge q))] \vee (p \rightarrow q)$ and $(q \rightarrow p) \wedge (p \rightarrow q)$ differ, not equivalent.

Answer: Not equivalent.

2. (c) Use laws of algebra of propositions to simplify $[p \wedge (p \vee q)] \vee [q \wedge (\neg(p \wedge q))]$.

$[p \wedge (p \vee q)] \vee [q \wedge (\neg(p \wedge q))]$

$= p \wedge (p \vee q) \vee (q \wedge (\neg p \vee \neg q))$ (De Morgan's)

$= p \vee (q \wedge \neg p \wedge \neg q)$ (Distributive, Absorption)

$= p \vee (q \wedge \neg q \wedge \neg p)$ (Commutative)

$= p \vee (0 \wedge \neg p)$ (Complement)

$= p \vee 0 = p$

Answer: p.

3. (a) Find the work done by force $F = i + 2j + k$ moving an object at a distance of 7 m in the direction of the vector $r = 3i + 2j + 4k$.

Unit vector in direction of r: $|r| = \sqrt{(3^2 + 2^2 + 4^2)} = \sqrt{(9 + 4 + 16)} = \sqrt{29}$

$$\text{Unit vector} = (3/\sqrt{29}, 2/\sqrt{29}, 4/\sqrt{29})$$

$$\text{Force component: } F \cdot (r/|r|) = (1 \times 3/\sqrt{29}) + (2 \times 2/\sqrt{29}) + (1 \times 4/\sqrt{29}) = (3 + 4 + 4) / \sqrt{29} = 11 / \sqrt{29}$$

$$\text{Work} = (11 / \sqrt{29}) \times 7 = 77 / \sqrt{29}$$

Answer: Work done = $77 / \sqrt{29}$ units.

3. (b) If P and Q are the points P(3, -4, 6) and Q(1, -3, 8) respectively, find a unit vector parallel to the displacement vector PQ.

$$\text{PQ} = \text{Q} - \text{P} = (1 - 3, -3 - (-4), 8 - 6) = (-2, 1, 2)$$

$$|\text{PQ}| = \sqrt{(-2)^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$

$$\text{Unit vector} = (-2/3, 1/3, 2/3)$$

Answer: Unit vector = $(-2/3, 1/3, 2/3)$.

3. (c) The position vectors of points A and B are a and b respectively. If point C divides AB internally in the ratio of 2:1, D divides AB externally in the ratio of 1:4 and E divides CD internally in the ratio of 2:1, find the position vectors of C, D, and E in terms of a and b.

$$\text{C divides AB in 2:1: } C = (1a + 2b) / (2 + 1) = (a + 2b) / 3$$

$$\text{D divides AB externally in 1:4: } D = (1b - 4a) / (1 - 4) = (-4a + b) / (-3) = (4a - b) / 3$$

$$\text{E divides CD in 2:1: } E = (1C + 2D) / (1 + 2) = ((a + 2b)/3 + 2(4a - b)/3) / 3 = (a + 2b + 8a - 2b) / 9 = (9a / 9) = a$$

Answer: $C = (a + 2b) / 3$, $D = (4a - b) / 3$, $E = a$.

4. (a) Express $\sqrt[4]{1 + i}$ in polar form.

$$1 + i: r = \sqrt{1^2 + 1^2} = \sqrt{2}, \theta = \tan^{-1}(1/1) = \pi/4$$

$$1 + i = \sqrt{2} (\cos(\pi/4) + i \sin(\pi/4))$$

$$\sqrt[4]{1 + i} = (\sqrt{2})^{1/2} (\cos(\pi/4 + 2k\pi)/2 + i \sin(\pi/4 + 2k\pi)/2)$$

$$= 2^{1/4} (\cos(\pi/8) + i \sin(\pi/8)) \quad (k = 0)$$

$$= 2^{1/4} (\cos(5\pi/8) + i \sin(5\pi/8)) \quad (k = 1)$$

Answer: $2^{1/4} (\cos(\pi/8) + i \sin(\pi/8))$, $2^{1/4} (\cos(5\pi/8) + i \sin(5\pi/8))$.

4. (b) Using the results in part (a), show that $\tan(\pi/8) = \sqrt{2} - 1$.

$$\sqrt[4]{1 + i} = 2^{1/4} (\cos(\pi/8) + i \sin(\pi/8))$$

Also, $\sqrt[4]{(1+i)} = (1+i)^{1/2}$, let $(1+i)^{1/2} = a + bi$

$$(1+i) = (a+bi)^2 = a^2 - b^2 + 2abi$$

$$\text{Real: } a^2 - b^2 = 1$$

$$\text{Imaginary: } 2ab = 1$$

$$\text{Solve: } ab = 1/2, b = 1/(2a), \text{ substitute: } a^2 - (1/(2a))^2 = 1$$

$$a^2 - 1/(4a^2) = 1$$

$$4a^4 - 1 = 4a^2$$

$$4a^4 - 4a^2 - 1 = 0$$

$$a^2 = (4 \pm \sqrt{(16+16)}) / 8 = (4 \pm 4\sqrt{2}) / 8 = (1 \pm \sqrt{2}) / 2$$

$$a^2 = (1 + \sqrt{2}) / 2, a = \sqrt{(1 + \sqrt{2})/2}$$

$$b = 1/(2a) = \sqrt{((\sqrt{2} - 1)/2)}$$

$$\tan(\pi/8) = b/a = \sqrt{((\sqrt{2} - 1)/(1 + \sqrt{2}))} = \sqrt{2} - 1 \text{ (after rationalizing)}$$

$$\text{Answer: } \tan(\pi/8) = \sqrt{2} - 1.$$

4. (c) If $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, prove that $\text{Arg}(z_1 / z_2) = \text{Arg}(z_1) - \text{Arg}(z_2)$.

$$z_1 / z_2 = (r_1 / r_2) (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

$$\text{Arg}(z_1 / z_2) = \theta_1 - \theta_2 = \text{Arg}(z_1) - \text{Arg}(z_2)$$

Answer: Proven.

4. (d) The complex numbers $z_1 = c / (1 + i)$ and $z_2 = d / (1 + 2i)$ where $c, d \in \mathbb{R}$ are such that $z_1 + z_2 = 1$, find the values of c and d .

$$z_1 = c(1 - i) / (1 + i)(1 - i) = c(1 - i) / 2$$

$$z_2 = d(1 - 2i) / (1 + 2i)(1 - 2i) = d(1 - 2i) / 5$$

$$z_1 + z_2 = (c/2 - ci/2) + (d/5 - 2di/5) = 1$$

$$\text{Real: } c/2 + d/5 = 1$$

$$\text{Imaginary: } -c/2 - 2d/5 = 0$$

$$\text{From imaginary: } -c/2 = 2d/5, c = -4d/5$$

Substitute: $(-4d/5)/2 + d/5 = 1$

$$-2d/5 + d/5 = 1$$

$$-d/5 = 1$$

$$d = -5$$

$$c = -4(-5)/5 = 4$$

Verify: $z_1 = 4(1 - i)/2 = 2 - 2i$, $z_2 = -5(1 - 2i)/5 = -1 + 2i$, $z_1 + z_2 = 1$

Answer: $c = 4$, $d = -5$.

5. (a) For all values of α , show that $\sin 3\alpha - \cos 3\alpha = 2$.

This appears incorrect as $\sin 3\alpha - \cos 3\alpha \neq 2$ for all α (e.g., $\alpha = 0$: $\sin 0 - \cos 0 = 0 - 1 = -1$). Likely a typo; let's assume the intended problem is to evaluate or prove a trigonometric identity. Let's try a similar identity or rephrase. Assume $\sin 3\alpha - \cos 3\alpha = 2 \sin \alpha \cos \alpha$ (as a possible correction).

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$$

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$$

$$\sin 3\alpha - \cos 3\alpha = (3 \sin \alpha - 4 \sin^3 \alpha) - (4 \cos^3 \alpha - 3 \cos \alpha)$$

This does not simplify to 2, so the problem statement is likely incorrect. Without a correct identity, we cannot proceed. Let's move to the next problem.

Answer: Problem statement appears incorrect; cannot be shown as given.

5. (b) Prove that $(\sin x + \sin 2x + \sin 3x) / (\cos x + \cos 2x + \cos 3x) = \tan 2x$.

Numerator: $\sin x + \sin 3x + \sin 2x$

$$\sin x + \sin 3x = 2 \sin((x + 3x)/2) \cos((x - 3x)/2) = 2 \sin 2x \cos(-x) = 2 \sin 2x \cos x$$

$$\text{Add } \sin 2x: 2 \sin 2x \cos x + \sin 2x = \sin 2x (2 \cos x + 1)$$

Denominator: $\cos x + \cos 3x + \cos 2x$

$$\cos x + \cos 3x = 2 \cos((x + 3x)/2) \cos((x - 3x)/2) = 2 \cos 2x \cos x$$

$$\text{Add } \cos 2x: 2 \cos 2x \cos x + \cos 2x = \cos 2x (2 \cos x + 1)$$

$$\text{Expression: } [\sin 2x (2 \cos x + 1)] / [\cos 2x (2 \cos x + 1)] = \sin 2x / \cos 2x = \tan 2x$$

Answer: Proven.

5. (c) Solve for β in the trigonometric equation $\tan^{-1}((\beta - 2)/(\beta - 2)) + \tan^{-1}((\beta + 1)/(\beta + 2)) = \pi/4$.

Since $(\beta - 2)/(\beta - 2) = 1$ for $\beta \neq 2$, first term is $\tan^{-1}(1) = \pi/4$.

Equation: $\pi/4 + \tan^{-1}((\beta + 1)/(\beta + 2)) = \pi/4$

$$\tan^{-1}((\beta + 1)/(\beta + 2)) = 0$$

$$(\beta + 1)/(\beta + 2) = 0$$

$$\beta + 1 = 0$$

$$\beta = -1$$

Check: $\tan^{-1}((-1 - 2)/(-1 - 2)) + \tan^{-1}((-1 + 1)/(-1 + 2)) = \pi/4 + \tan^{-1}(0) = \pi/4$, correct.

Answer: $\beta = -1$.

5. (d) Rewrite $4 \cos \theta - 3 \sin \theta$ in the form $R \cos(\theta - \alpha)$, hence solve the equation $4 \cos \theta - 3 \sin \theta = 5/2$ in the interval $\pi/2 \leq \theta \leq 2\pi$.

$$R = \sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$\cos \alpha = 4/5, \sin \alpha = -3/5$$

$$4 \cos \theta - 3 \sin \theta = 5 (\cos \theta \cos \alpha + \sin \theta \sin \alpha) = 5 \cos(\theta - \alpha)$$

$$5 \cos(\theta - \alpha) = 5/2$$

$$\cos(\theta - \alpha) = 1/2$$

$$\theta - \alpha = \pm\pi/3 + 2k\pi$$

$$\alpha = \arccos(4/5), \theta = \pm\pi/3 + \alpha + 2k\pi$$

For $\pi/2 \leq \theta \leq 2\pi$, $k = 0$:

$$\theta = \pi/3 + \alpha (\approx 1.97, \text{ within range})$$

$$\theta = -\pi/3 + \alpha + 2\pi (\approx 5.71, \text{ within range})$$

Answer: $\theta = \pi/3 + \alpha, 2\pi - \pi/3 + \alpha$, where $\cos \alpha = 4/5$.

6. (a) If the coefficients of x and x^2 in the expansion of $(1 + px + qx^2) / (1 - x)^2$ are zero, find the numerical values of p and q .

Expand $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$ (binomial expansion for $n = -2$)

$$(1 + px + qx^2) / (1 - x)^2 = (1 + px + qx^2) (1 + 2x + 3x^2 + 4x^3 + \dots)$$

$$= 1 + (p + 2)x + (q + 2p + 3)x^2 + (4p + 3q + 4)x^3 + \dots$$

Coefficient of x : $p + 2 = 0$, $p = -2$

Coefficient of x^2 : $q + 2p + 3 = 0$, $q + 2(-2) + 3 = 0$, $q - 4 + 3 = 0$, $q = 1$

Answer: $p = -2$, $q = 1$.

6. (b) Use the principle of mathematical induction to prove that for every positive integer, $3^{(2n-2)} + 2^{(6n)}$ is divisible by 5.

Base case ($n = 1$):

$$3^{(2(1)-2)} + 2^{(6(1))} = 3^0 + 2^6 = 1 + 64 = 65, 65 / 5 = 13, \text{ divisible by } 5.$$

Assume true for $n = k$: $3^{(2k-2)} + 2^{(6k)} = 5m$

For $n = k + 1$:

$$3^{(2(k+1)-2)} + 2^{(6(k+1))} = 3^{(2k)} + 2^{(6k+6)}$$

$$= 9 \times 3^{(2k-2)} + 64 \times 2^{(6k)}$$

$$= 9(5m - 2^{(6k)}) + 64 \times 2^{(6k)} \text{ (using inductive hypothesis)}$$

$$= 45m - 9 \times 2^{(6k)} + 64 \times 2^{(6k)}$$

$$= 45m + 55 \times 2^{(6k)}$$

$45m$ is divisible by 5, $55 \times 2^{(6k)}$ is divisible by 5 (since $55 / 5 = 11$), so the sum is divisible by 5.

Answer: Proven, $3^{(2n-2)} + 2^{(6n)}$ is divisible by 5 for all positive integers n .

6. (c) Given that $P(x) = 2x^3 + 7x^2 - 5$, use the synthetic method to find the quotient and remainder when $P(x)$ is divided by $x + 3$.

Synthetic division: $x + 3 \rightarrow x = -3$

$$\begin{array}{r|rrrr} -3 & 2 & 7 & -5 & \end{array}$$

$$\begin{array}{r|rrrr} & -6 & -3 & 24 & \end{array}$$

$$\begin{array}{r|rrrr} & 2 & 1 & -8 & 24 \end{array}$$

Quotient: $2x^2 + x - 8$

Remainder: 24

Answer: Quotient = $2x^2 + x - 8$, Remainder = 24.

6. (d) Find the determinant and inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, hence solve the simultaneous equations:

$$2x + y = 4$$

$$x + 5y + 2z = 7$$

$$x - y + z = 1$$

$$\det(A) = 2(5 \times 1 - 2 \times (-1)) - 1(1 \times 1 - 2 \times 1) + 0 = 2(5 + 2) - 1(1 - 2) = 14 + 1 = 15$$

Cofactors:

$$C_{11} = 5 \times 1 - 2 \times (-1) = 7, C_{12} = -(1 \times 1 - 2 \times 1) = 1, C_{13} = (1 \times (-1) - 5 \times 1) = -6$$

$$C_{21} = -(1 \times 1 - 0 \times (-1)) = -1, C_{22} = (2 \times 1 - 0 \times 1) = 2, C_{23} = -((2 \times (-1) - 1 \times 0)) = 2$$

$$C_{31} = (1 \times 2 - 5 \times 0) = 2, C_{32} = -((2 \times 2 - 0 \times 1)) = -4, C_{33} = (2 \times 5 - 1 \times 1) = 9$$

$$\text{Adjoint: } \begin{bmatrix} 7 & -1 & 2 \\ 1 & 2 & -4 \\ -6 & 2 & 9 \end{bmatrix}$$

$$A^{-1} = (1/15) \begin{bmatrix} 7 & -1 & 2 \\ 1 & 2 & 2 \\ 2 & -4 & 9 \end{bmatrix}$$

$$\text{Solve: } X = A^{-1} B, B = [4, 7, 1]$$

$$x = (1/15)(7 \times 4 + 1 \times 7 + (-6) \times 1) = (1/15)(28 + 7 - 6) = 39/15 = 13/5$$

$$y = (1/15)(-1 \times 4 + 2 \times 7 + 2 \times 1) = (1/15)(-4 + 14 + 2) = 12/15 = 4/5$$

$$z = (1/15)(2 \times 4 + (-4) \times 7 + 9 \times 1) = (1/15)(8 - 28 + 9) = -11/15$$

$$\text{Answer: } \det(A) = 15, A^{-1} = (1/15) \begin{bmatrix} 7 & -1 & 2 \\ 1 & 2 & 2 \\ 2 & -4 & 9 \end{bmatrix}, x = 13/5, y = 4/5, z = -11/15.$$

7. (a) (i) Determine the most general function $M(x, y)$ such that the differential equation $M(x, y) dx + (2x^2 y^2 + x^2 y) dy = 0$ is exact.

$$\text{For exactness: } \partial M / \partial y = \partial(2x^2 y^2 + x^2 y) / \partial x = 4x y^2 + 2x y$$

$$M = \int (4x y^2 + 2x y) dy = 4x y^3/3 + x y^2 + f(x)$$

$$\text{Answer: } M(x, y) = 4x y^3/3 + x y^2 + f(x).$$

7. (a) (ii) By separating the variables, solve the differential equation $(x + y^2) dx - (x^2 y^2 + x + y + 1) dy = 0$.

We are solving the differential equation:

$$(xy + x) dx - (x^2 y^2 + x^2 + y^2 + 1) dy = 0$$

Write it as $M(x, y) dx + N(x, y) dy = 0$

$$M(x, y) = x(y + 1)$$

$$N(x, y) = -(x^2y^2 + x^2 + y^2 + 1)$$

Rewriting the equation:

$$dy/dx = (x(y + 1)) / (x^2y^2 + x^2 + y^2 + 1)$$

Factor the denominator

$$x^2y^2 + x^2 + y^2 + 1$$

$$= x^2(y^2 + 1) + (y^2 + 1)$$

$$= (y^2 + 1)(x^2 + 1)$$

Substitute the factorized denominator

$$dy/dx = (x(y + 1)) / ((y^2 + 1)(x^2 + 1))$$

Step 4: Separate variables

Multiply both sides by $(y^2 + 1) dy$ and multiply both sides by dx denominator to isolate variables

$$(y^2 + 1) dy = (x(y + 1)) / (x^2 + 1) dx$$

Now, move all terms involving y to one side and x to the other

$$(y^2 + 1) / (y + 1) dy = (x) / (x^2 + 1) dx$$

Simplify left side

Divide $(y^2 + 1)$ by $(y + 1)$ using polynomial division or factor if possible.

Let's divide:

$$(y^2 + 1) \div (y + 1)$$

$$= y^2 + 1 = (y + 1)(y - 1) + 2$$

So:

$$(y^2 + 1) / (y + 1) = y - 1 + 2 / (y + 1)$$

Now the separated equation is:

$$(y - 1 + 2 / (y + 1)) dy = (x) / (x^2 + 1) dx$$

Integrate both sides

$$\int (y - 1) dy + \int (2 / (y + 1)) dy$$

$$= (1/2) y^2 - y + 2 \ln|y + 1|$$

Integrating right side:

$$\int x / (x^2 + 1) dx$$

Let $u = x^2 + 1$, then $du = 2x dx$

So, $(1/2) du = x dx$

$$\int x / (x^2 + 1) dx = (1/2) \int du / u = (1/2) \ln|x^2 + 1|$$

$$(1/2) y^2 - y + 2 \ln|y + 1| = (1/2) \ln|x^2 + 1| + C$$

7. (b) Find the general solution of the differential equation $\cos x \, d^2y/dx^2 - \sin x \, dy/dx = 0$.

$$\cos x \, d^2y/dx^2 - \sin x \, dy/dx = 0$$

$$d^2y/dx^2 - (\sin x / \cos x) dy/dx = 0$$

Let $u = dy/dx$:

$$du/dx - (\tan x) u = 0$$

$$du/u = \tan x \, dx$$

$$\ln|u| = -\ln|\cos x| + C$$

$$u = K / \cos x$$

$$dy/dx = K / \cos x$$

$$y = K \int (1/\cos x) dx + D = K \ln|\tan(x/2 + \pi/4)| + D$$

Answer: $y = K \ln|\tan(x/2 + \pi/4)| + D$.

7. (c) A liquid of 72°C placed in a room at 25°C has a temperature of 65°C after 5 minutes. Find its temperature after further 10 minutes.

$$dT/dt = -k (T - 25)$$

$$T - 25 = C e^{(-kt)}$$

$$\text{At } t = 0, T = 72: 72 - 25 = C, C = 47$$

$$\text{At } t = 5, T = 65: 65 - 25 = 47 e^{(-5k)}, 40/47 = e^{(-5k)}$$

$$\ln(40/47) = -5k, k = -(1/5) \ln(40/47)$$

$$\text{At } t = 15 \text{ (further 10 minutes): } T - 25 = 47 (40/47)^{(15/5)} = 47 (40/47)^3$$

$$(40/47)^3 \approx 0.617$$

$$T - 25 = 47 \times 0.617 \approx 29$$

$$T \approx 54$$

Answer: Temperature $\approx 54^\circ\text{C}$.

7. (d) Formulate a differential equation of a circle which passes through the origin and whose centre lies on the y-axis.

Centre $(0, k)$, passes through $(0, 0)$:

$$x^2 + (y - k)^2 = k^2$$

$$x^2 + y^2 - 2ky = 0$$

$$\text{Differentiate: } 2x + 2y \, dy/dx - 2k \, dy/dx = 0$$

$$x + y \, dy/dx - k \, dy/dx = 0$$

$$k = (x + y \, dy/dx) / (dy/dx)$$

Substitute into original: $x^2 + y^2 - 2(x + y \, dy/dx) / (dy/dx) y = 0$ (simplify as needed).

Answer: $x^2 + y^2 - 2(x + y \, dy/dx) / (dy/dx) y = 0$.

8. (a) Find the coordinates of the foci, the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $16x^2 - 9y^2 = 576$.

$$16x^2 - 9y^2 = 576$$

$$x^2/36 - y^2/64 = 1$$

$$a = 6, b = 8$$

$$\text{Foci: } (\pm\sqrt{a^2 + b^2}, 0) = (\pm\sqrt{36 + 64}, 0) = (\pm 10, 0)$$

Vertices: $(\pm a, 0) = (\pm 6, 0)$

Eccentricity: $e = \sqrt{(a^2 + b^2)} / a = 10/6 = 5/3$

Latus rectum: $2b^2/a = 2 \times 64 / 6 = 128/6 = 64/3$

Answer: Foci: $(\pm 10, 0)$, Vertices: $(\pm 6, 0)$, Eccentricity: $5/3$, Latus rectum: $64/3$.

8. (b) (i) Determine the equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $P(a \cos \theta, b \sin \theta)$.

$$dy/dx = -(b^2 x) / (a^2 y), \text{ at } P: dy/dx = -(b^2 a \cos \theta) / (a^2 b \sin \theta) = -(b/a) \cot \theta$$

Normal slope = $(a/b) \tan \theta$

Normal: $y - b \sin \theta = (a/b) \tan \theta (x - a \cos \theta)$

Simplify: $(a x \sin \theta) / (\cos \theta) - (b y \cos \theta) / (\sin \theta) = a^2 - b^2$

Answer: $(a x \sin \theta) / (\cos \theta) - (b y \cos \theta) / (\sin \theta) = a^2 - b^2$.

8. (b) (ii) If the normal in part (b) (i) meets the x-axis at A and the y-axis at B, find the area of the triangle AOB where O is the origin.

A: $((a^2 - b^2) \cos \theta / a, 0)$

B: $(0, (b^2 - a^2) \sin \theta / b)$

$$\text{Area} = (1/2) | (a^2 - b^2) \cos \theta / a \times (b^2 - a^2) \sin \theta / b | = (a^2 - b^2)^2 |\sin \theta \cos \theta| / (2ab)$$

Answer: $\text{Area} = (a^2 - b^2)^2 |\sin \theta \cos \theta| / (2ab)$.

8. (c) Show that the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is $x - ty + at^2 = 0$.

$$y^2 = 4ax, dy/dx = 2a/y, \text{ at } (at^2, 2at): dy/dx = 2a / (2at) = 1/t$$

Tangent: $y - 2at = (1/t) (x - at^2)$

$$ty - 2at^2 = x - at^2$$

$$x - ty + at^2 = 0$$

Answer: Proven.

8. (d) (i) Change the Cartesian equation $(x^2 + y^2)^2 - 2xy(x^2 - y^2) = 0$ into a polar equation.

$$x = r \cos \theta, y = r \sin \theta$$

$$(x^2 + y^2)^2 - 2xy(x^2 - y^2) = (r^2)^2 - 2(r \cos \theta \ r \sin \theta)(r^2 \cos^2 \theta - r^2 \sin^2 \theta) = r^4 - 2r^4 \cos \theta \sin \theta (\cos^2 \theta - \sin^2 \theta)$$

$$= r^4 (1 - 2 \sin \theta \cos \theta (\cos 2\theta)) = r^4 (1 - \sin 2\theta \cos 2\theta) = 0$$

$$r = 0 \text{ or } 1 - \sin 2\theta \cos 2\theta = 0 \text{ (simplify further if needed).}$$

$$\text{Answer: } r = 0 \text{ or } 1 - \sin 2\theta \cos 2\theta = 0.$$

8. (d) (ii) Sketch the graph of $r = 1 - 2 \cos \theta$ from $\theta = 0$ to $\theta = 2\pi$.

$r = 1 - 2 \cos \theta$ (limaçon with a loop):

$$\theta = 0: r = 1 - 2 = -1$$

$$\theta = \pi/2: r = 1 - 0 = 1$$

$$\theta = \pi: r = 1 + 2 = 3$$

$$\theta = 3\pi/2: r = 1 - 0 = 1$$

Graph: Limaçon with inner loop, symmetric about x-axis.

Answer: Limaçon with an inner loop.