

THE UNITED REPUBLIC OF TANZANIA
NATIONAL EXAMINATIONS COUNCIL
CERTIFICATE OF SECONDARY EDUCATION EXAMINATION
042
ADDITIONAL MATHEMATICS

(For Both School and Private Candidates)

Time: 3 Hours

ANSWERS

Year: 2001

Instructions

1. This paper consists of SIXTEEN questions.
2. Answer all questions in section A and any FOUR questions in section B.

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1. (a) The functions f and g are defined by

$$f: x \rightarrow 5x + 4$$

$$g: x \rightarrow 6x - k$$

where $x \in \mathbb{R}$ and k is a constant.

Find the value of k for which $\text{fog}(x) = \text{gof}(x)$.

$$\text{fog}(x) = f(g(x)) = f(6x - k) = 5(6x - k) + 4 = 30x - 5k + 4$$

$$\text{gof}(x) = g(f(x)) = g(5x + 4) = 6(5x + 4) - k = 30x + 24 - k$$

Now equate:

$$30x - 5k + 4 = 30x + 24 - k$$

Cancel $30x$:

$$-5k + 4 = 24 - k$$

$$\text{Rearrange: } -5k + k = 24 - 4 \rightarrow -4k = 20 \rightarrow k = -5$$

(b) The expression $3x^3 + 2x^2 - bx + a$ is divisible by $(x - 1)$ and leaves a remainder of 9 when divided by $(x + 1)$. Find the values of a and b .

$$\text{Let } f(x) = 3x^3 + 2x^2 - bx + a$$

If $(x - 1)$ is a factor, then $f(1) = 0$

$$f(1) = 3 + 2 - b + a = 0 \rightarrow a - b = -5 \dots(i)$$

Also, $f(-1) = 9$

$$f(-1) = -3 + 2 + b + a = 9 \rightarrow a + b = 10 \dots(ii)$$

Add (i) and (ii):

$$a - b + a + b = -5 + 10 \rightarrow 2a = 5 \rightarrow a = 2.5$$

$$\text{Then } b = 10 - 2.5 = 7.5$$

2. (a) Prove that $(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) = \sin^3\theta + \cos^3\theta$.

Left-hand side:

$$(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$$

$$= \sin\theta(1 - \sin\theta\cos\theta) + \cos\theta(1 - \sin\theta\cos\theta)$$

$$= \sin\theta - \sin^2\theta\cos\theta + \cos\theta - \sin\theta\cos^2\theta$$

$$= \sin\theta + \cos\theta - \sin^2\theta\cos\theta - \sin\theta\cos^2\theta$$

Group terms:

$$= \sin\theta + \cos\theta - \sin\theta\cos\theta(\sin\theta + \cos\theta)$$

$$= (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) \text{ --- confirmed equal to LHS}$$

Now RHS:

$$\sin^3\theta + \cos^3\theta$$

$$\text{Use identity: } a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta)$$

$$= (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) \text{ --- same as LHS}$$

So identity is proven.

(b) Find the value of $\sin^2 A \operatorname{cosec}^2(\pi/2 - A) - \cot^2(\pi/2 - A)\cos^2 A$

We use identities:

$$\operatorname{cosec}(\pi/2 - A) = \sec A \rightarrow \operatorname{cosec}^2(\pi/2 - A) = \sec^2 A$$

$$\cot(\pi/2 - A) = \tan A \rightarrow \cot^2(\pi/2 - A) = \tan^2 A$$

Now expression becomes:

$$\sin^2 A \times \sec^2 A - \tan^2 A \times \cos^2 A$$

$$\sin^2 A \times (1/\cos^2 A) - (\sin^2 A/\cos^2 A) \times \cos^2 A$$

$$= \sin^2 A/\cos^2 A - \sin^2 A = \tan^2 A - \sin^2 A$$

Final simplified result: $\tan^2 A - \sin^2 A$

3. If $u = 2i - j$, $v = 6i - 3j$

(i) Find $u + v$

$$u + v = (2 + 6)i + (-1 - 3)j = 8i - 4j$$

(ii) Show that $|u + v| = |u| + |v|$

$$|u + v| = \sqrt{(8)^2 + (-4)^2} = \sqrt{(64 + 16)} = \sqrt{80}$$

$$|u| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$$

$$|v| = \sqrt{(6)^2 + (-3)^2} = \sqrt{45}$$

$$|u| + |v| = \sqrt{5} + \sqrt{45}$$

Since $\sqrt{80} \neq \sqrt{5} + \sqrt{45} \rightarrow$ statement is false

4. (a) In a triangle OAB, $OA = a$ and $OB = b$. Given that P and Q are the midpoints of OA and OB respectively, express vector PQ and AB in terms of a and b.

$$OP = \frac{1}{2}a, OQ = \frac{1}{2}b$$

$$PQ = OQ - OP = \frac{1}{2}b - \frac{1}{2}a = \frac{1}{2}(b - a)$$

$$AB = b - a$$

$$\text{So } PQ = \frac{1}{2} AB$$

Relationship: PQ is parallel to AB and half its length

(b) State the geometrical relationship between PQ and AB.

They are parallel and $PQ = \frac{1}{2} AB$

5. Calculate the median and upper quartile of the distribution.

Mass (g)	Frequency	Cumulative Freq
75–79	7	7

80–84	30	37	
85–89	66	103	
90–94	57	160	
95–99	27	187	
100–104	13	200	

Median class = $(200/2 = 100) \rightarrow$ class 85–89

Median = $L + [(n/2 - F)/f] \times h$

$L = 84.5, F = 37, f = 66, h = 5$

Median = $84.5 + [(100 - 37)/66] \times 5 = 84.5 + (63/66) \times 5 \approx 84.5 + 4.77 = 89.27g$

Upper quartile: $\frac{3}{4} \times 200 = 150 \rightarrow$ class 90–94

$L = 89.5, F = 103, f = 57$

$Q_3 = 89.5 + [(150 - 103)/57] \times 5 = 89.5 + (47/57) \times 5 \approx 89.5 + 4.12 = 93.62g$

6. A social committee is to transport 20 boys and 32 girls to a place for a picnic.

The committee can hire either a taxi (2 boys + 1 girl) or a mini bus (2 boys + 4 girls).

Costs: taxi = 4000, mini bus = 3000

Find the cheapest means of transport.

Let $x =$ no. of taxis, $y =$ no. of mini buses

$2x + 2y = 20 \rightarrow x + y = 10$

$x + 4y = 32$

From above: $y = 10 - x$

Substitute: $x + 4(10 - x) = 32 \rightarrow x + 40 - 4x = 32 \rightarrow -3x = -8 \rightarrow x = 8/3$

x not whole number, try trial:

$x = 4 \rightarrow y = 6 \rightarrow 2x + 2y = 8 + 12 = 20$ boys, $x + 4y = 4 + 24 = 28$ girls

Not 32

Try $x = 2 \rightarrow y = 8$

$2x + 2y = 4 + 16 = 20$ boys

$x + 4y = 2 + 32 = 34$ girls $\rightarrow$ exceeds

Try $x = 3, y = 7 \rightarrow 6 + 14 = 20$ boys, $3 + 28 = 31$ girls $\rightarrow$ too small

Try $x = 4, y = 6 \rightarrow 8 + 12 = 20$ boys, $4 + 24 = 28$ girls

Only $x = 6, y = 5 \rightarrow 12 + 10 = 22$ boys, $6 + 20 = 26$ girls

Try $x = 5, y = 5 \rightarrow 10 + 10 = 20$ boys, $5 + 20 = 25$ girls

Only valid: $x = 4, y = 7 \rightarrow 8 + 14 = 22$ boys, $4 + 28 = 32$ girls

Finally: $x = 2, y = 8 \rightarrow$ costs = $2 \times 4000 + 8 \times 3000 = 8000 + 24000 = 32000$

Cheapest cost = 32000

7. (a) Differentiate with respect to x :

(i) $y = \sin x / (1 + \tan x)$

Use quotient rule:

$$dy/dx = [\cos x(1 + \tan x) - \sin x(\sec^2 x)] / (1 + \tan x)^2$$

$$(ii) y = \sqrt{x^2 + 2x}$$

$$\text{Let } u = x^2 + 2x$$

$$dy/dx = \frac{1}{2}(x^2 + 2x)^{-1/2} \times (2x + 2) = (x + 1)/\sqrt{x^2 + 2x}$$

$$\begin{aligned} (b) \text{ Evaluate } \int_0^{\pi/2} (2\cos^2\theta + 3\sin^2\theta) d\theta \\ &= \int_0^{\pi/2} [2(1 + \cos 2\theta)/2 + 3(1 - \cos 2\theta)/2] d\theta \\ &= \int_0^{\pi/2} [2/2 + 2\cos 2\theta/2 + 3/2 - 3\cos 2\theta/2] \\ &= \int_0^{\pi/2} [(5/2) - (\cos 2\theta)/2] \\ &= 5\theta/2 - \sin 2\theta/4 \text{ from } 0 \text{ to } \pi/2 \\ &= 5\pi/4 - 0 = 5\pi/4 \end{aligned}$$

8. (a) Show how the biconditional $p \leftrightarrow q$ can be written in terms of original three connectives $\vee, \wedge, \neg$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

(b) Using truth table verify that:

$$(i) \neg(p \rightarrow q) \equiv p \wedge \neg q$$

$$(ii) \neg(p \leftrightarrow q) \equiv p \oplus q \text{ (exclusive or)}$$

Use full 2-variable truth table to verify both expressions

$$9. (a) \int e^x \cos 3x dx$$

Use integration by parts twice or known result:

$$= (e^x/10)(\cos 3x + 3 \sin 3x) + C$$

(b) Find p and q such that

$$\int_0^x (pt - q) dt = 9x^2 + 9x$$

$$\text{Integrate: } (pt - q) \rightarrow p \times t^2/2 - qt$$

Apply limits:

$$p x^2/2 - qx = 9x^2 + 9x$$

$$\text{So: } p/2 = 9 \rightarrow p = 18$$

$$-q = 9 \rightarrow q = -9$$

10. Solve the following system of simultaneous equations:

$$2x - 5y + 2z = 14$$

$$9x + 3y - 4z = 13$$

$$7x + 3y - 2z = 3$$

Given,

$$2x - 5y + 2z = 14 \dots\dots\dots (1)$$

$$9x + 3y - 4z = 13 \dots\dots\dots (2)$$

$$7x + 3y - 2z = 3 \dots\dots\dots (3)$$

Step 1: Eliminate y from equations (2) and (3)

We can subtract (2) - (3) directly because both have 3y:

$$(2) - (3):$$

$$(9x - 7x) + (3y - 3y) + (-4z + 2z) = 13 - 3$$

$$\rightarrow 2x - 2z = 10 \dots\dots\dots (4)$$

Step 2: Eliminate y from equations (1) and (2)

We want to eliminate y, so we find LCM of coefficients of y in (1) and (2): LCM of 5 and 3 is 15

Multiply (1) by 3:

$$3(2x - 5y + 2z) = 6x - 15y + 6z = 42 \dots\dots\dots (5)$$

Multiply (2) by 5:

$$5(9x + 3y - 4z) = 45x + 15y - 20z = 65 \dots\dots\dots (6)$$

Now add (5) and (6):

$$(6x + 45x) + (-15y + 15y) + (6z - 20z) = 42 + 65$$

$$51x - 14z = 107 \dots\dots\dots (7)$$

Step 3: Use equations (4) and (7) to eliminate another variable

We now solve equations (4) and (7):

$$(4): 2x - 2z = 10$$

$$(7): 51x - 14z = 107$$

Multiply (4) by 7 to make coefficients of z equal ($2 \times 7 = 14$):

$$7(2x - 2z) = 14x - 14z = 70 \dots\dots\dots (8)$$

Now subtract (8) from (7):

$$(51x - 14z) - (14x - 14z) = 107 - 70$$

$$51x - 14x = 37x$$

$$\rightarrow 37x = 37$$

$$\rightarrow x = 1$$

Step 4: Substitute $x = 1$ into equation (4) to find z

From (4):

$$2x - 2z = 10$$

$$2(1) - 2z = 10 \rightarrow 2 - 2z = 10 \rightarrow -2z = 8 \rightarrow z = -4$$

Step 5: Substitute $x = 1$, $z = -4$ into equation (1) to find y

From (1):

$$2x - 5y + 2z = 14$$

$$2(1) - 5y + 2(-4) = 14$$

$$2 - 5y - 8 = 14 \rightarrow -5y - 6 = 14 \rightarrow -5y = 20 \rightarrow y = -4$$

Final solution:

$$x = 1, y = -4, z = -4$$

11. (a) Find the 2×2 matrix that will transform the point $(1, 2)$ to $(3, 3)$ and the point $(-1, 1)$ to $(-3, 3)$

Let the matrix be:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We want:

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix} \text{ and } \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \end{bmatrix}$$

$$\begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix} \quad \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix}$$

So the system:

$$a(1) + b(2) = 3 \rightarrow a + 2b = 3 \dots (1)$$

$$c(1) + d(2) = 3 \rightarrow c + 2d = 3 \dots (2)$$

$$a(-1) + b(1) = -3 \rightarrow -a + b = -3 \dots (3)$$

$$c(-1) + d(1) = 3 \rightarrow -c + d = 3 \dots (4)$$

From (1) and (3):

$$a + 2b = 3$$

$$-a + b = -3 \rightarrow \text{add} \rightarrow 3b = 0 \rightarrow b = 0$$

$$\text{Then } a = 3$$

From (2) and (4):

$$c + 2d = 3$$

$$-c + d = 3 \rightarrow \text{add} \rightarrow 3d = 6 \rightarrow d = 2$$

$$\text{Then } c = 3 - 2d = 3 - 4 = -1$$

So matrix is:

$$\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

(b) All the points on the line $y = 2x - 3$ have been transformed by the matrix

$$\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \end{bmatrix}$$

Find the equation of the image line.

Let (x, y) be on the line $y = 2x - 3$

Then image:

$$x' = 2x + y$$

$$y' = 3x - y$$

But $y = 2x - 3 \rightarrow$ substitute:

$$x' = 2x + (2x - 3) = 4x - 3$$

$$y' = 3x - (2x - 3) = x + 3$$

Eliminate x :

$$\text{From } x' = 4x - 3 \rightarrow 4x = x' + 3 \rightarrow x = (x' + 3)/4$$

$$\text{Then } y' = (x' + 3)/4 + 3 = (x' + 3 + 12)/4 = (x' + 15)/4$$

$$\text{Equation of image line: } y' = (1/4)x' + 15/4$$

12. Find the equation of the circle which circumscribes the triangle with vertices $(1, 0)$, $(2, 1)$, and $(0, 2)$

Let $A(1, 0)$, $B(2, 1)$, $C(0, 2)$

Find perpendicular bisectors of AB and BC and find intersection (center)

Midpoint $AB: ((1+2)/2, (0+1)/2) = (1.5, 0.5)$, slope $AB = (1-0)/(2-1) = 1$

Perpendicular slope $= -1$, equation: $y - 0.5 = -1(x - 1.5)$

$$\rightarrow y = -x + 2$$

Midpoint $BC = ((2+0)/2, (1+2)/2) = (1, 1.5)$, slope $BC = (2-1)/(0-2) = -1/2$

Perpendicular slope $= 2$, equation: $y - 1.5 = 2(x - 1)$

$$\rightarrow y = 2x - 0.5$$

$$\text{Solve: } -x + 2 = 2x - 0.5 \rightarrow 3x = 2.5 \rightarrow x = 5/6$$

$$\text{Then } y = -5/6 + 2 = 7/6$$

Center $= (5/6, 7/6)$, use A to find radius:

$$r^2 = (1 - 5/6)^2 + (0 - 7/6)^2 = (1/6)^2 + (-7/6)^2 = 1/36 + 49/36 = 50/36$$

So equation:

$$(x - 5/6)^2 + (y - 7/6)^2 = 50/36$$

13. (a) A company wishes to pack food in closed cylindrical tins of volume $128\pi \text{ cm}^3$.

Find dimensions of minimum area.

$$V = \pi r^2 h = 128\pi \rightarrow h = 128/r^2$$

$$\text{Area } A = 2\pi r^2 + 2\pi r h$$

Substitute $h = 128/r^2$:

$$A = 2\pi r^2 + 2\pi r(128/r^2) = 2\pi r^2 + 256\pi/r$$

Differentiate A:

$$dA/dr = 4\pi r - 256\pi/r^2$$

$$\text{Set to 0: } 4\pi r = 256\pi/r^2 \rightarrow 4r^3 = 256 \rightarrow r^3 = 64 \rightarrow r = 4$$

$$\text{Then } h = 128/16 = 8$$

Dimensions: radius = 4 cm, height = 8 cm

(b) Volume generated by rotating region between $y = 5$ and $y = x^2 + 1$ about $y = 5$

$$y = 5 - (x^2 + 1) = 4 - x^2 \text{ is radius}$$

$$V = \pi \int \text{from } x = -2 \text{ to } 2 \text{ of } (4 - x^2)^2 dx$$

$$= \pi \int_{-2}^2 (16 - 8x^2 + x^4) dx = \pi [16x - 8x^3/3 + x^5/5]_{-2}^2$$

Compute definite integral:

$$\text{At } x = 2: 32 - 64/3 + 32/5 = 160/5 - 64/3 + 32/5$$

$$= (160 + 32)/5 = 192/5$$

$$\rightarrow 192/5 - 64/3$$

$$\text{Common denom} = 15 \rightarrow (576 - 320)/15 = 256/15$$

$$\text{Double it: final } V = 2 \times 256/15 \times \pi = 512\pi/15 \text{ cm}^3$$

$$14. (a) P(A) = 5/8, P(B') = 3/7 \rightarrow P(B) = 4/7$$

$$P(A \cap B) = P(A) \times P(B) = 5/8 \times 4/7 = 20/56 = 5/14$$

(b) Class of 30 boys

15 have bicycles

10 have motorbikes

4 have both

So:

$$\text{Bicycle only} = 15 - 4 = 11$$

$$\text{Motorbike only} = 10 - 4 = 6$$

$$\text{None} = 30 - (11 + 6 + 4) = 9$$

$$(i) \text{ Neither} = 9/30 = 3/10$$

$$(ii) \text{ Bicycle but no motorbike} = 11/30$$

$$15. (a) z = \frac{1}{2} - \sqrt{3}/2 i$$

$$\text{Modulus} = \sqrt{[(1/2)^2 + (\sqrt{3}/2)^2]} = \sqrt{(1/4 + 3/4)} = \sqrt{1} = 1$$

$$\text{Argument} = \tan^{-1}(-\sqrt{3}) = -\pi/3$$

$$(b) z = 1 + i$$

$$z^3 = (1 + i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i - 3 - i = -2 + 2i$$

$$\text{Hence } z^3 = -2 + 2i$$

$$p/(1 + z) = q/(1 + z^3) = 2i$$

$$1 + z = 1 + 1 + i = 2 + i$$

$$1 + z^3 = 1 - 2 + 2i = -1 + 2i$$

So:

$$p/(2 + i) = 2i \rightarrow p = 2i(2 + i) = 4i + 2i^2 = 4i - 2 = -2 + 4i$$

$$q/(-1 + 2i) = 2i \rightarrow q = 2i(-1 + 2i) = -2i + 4i^2 = -2i - 4 = -4 - 2i$$

$$(c) 2z - 4\bar{z} = 3 - 6i$$

$$\text{Let } z = x + iy \rightarrow \bar{z} = x - iy$$

$$2(x + iy) - 4(x - iy) = 3 - 6i$$

$$2x + 2iy - 4x + 4iy = 3 - 6i$$

$$-2x + 6iy = 3 - 6i$$

Compare real and imaginary:

$$-2x = 3 \rightarrow x = -1.5$$

$$6iy = -6i \rightarrow y = -1$$

$$\text{So } z = -1.5 - i$$

16. Solve $x^3 - x - 6 = 0$ between 1 and 2 using Newton-Raphson, starting $x_0 = 1.6$

$$f(x) = x^3 - x - 6$$

$$f'(x) = 3x^2 - 1$$

$$x_1 = x_0 - f(x_0)/f'(x_0)$$

$$f(1.6) = 4.096 - 1.6 - 6 = -3.504$$

$$f'(1.6) = 3(2.56) - 1 = 6.68$$

$$x_1 = 1.6 - (-3.504)/6.68 = 1.6 + 0.524 = 2.124$$

$$x_2 = 2.124 - f(2.124)/f'(2.124)$$

$$f(2.124) = 9.58 - 2.124 - 6 = 1.456$$

$$f' = 3(4.512) - 1 = 12.54$$

$$x_2 = 2.124 - (1.456)/12.54 \approx 2.008$$

Better root ≈ 2.01 to 2 d.p.